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A Comparative study between Boundary and Finite Element Techniques for solving Inverse Problems

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Abstract:

Inverse problems are of central importance in many scientific and engineering disciplines, where the main objective is to determine unknown internal properties based on limited or indirect observations. Among the most prominent numerical approaches used to address such problems are the Finite Element Method (FEM) and the Boundary Element Method (BEM). This paper presents a comparative analytical study between the two methods with respect to numerical accuracy, computational requirements, and the treatment of boundary conditions. The BEM has proven its efficiency in unbounded domains, as it requires discretization of boundaries only, which significantly reduces the problem size. On the other hand, the FEM remains more suitable for problems involving complex geometries, heterogeneous materials, and nonlinearities. The results highlight that the selection of the most appropriate method ultimately depends on the specific nature of the inverse problem, the type of governing equations, and the availability of reliable data.

Keyword : Finite Element Method (FEM), Boundary Element Method (BEM), Inverse Problems, Ill-posedness, Regularization Approaches

Introduction:

Inverse problems occupy a central role in various scientific and engineering applications, as they involve the reconstruction of unknown system parameters from observed or indirect data. Typically, such problems do not satisfy the criteria of well-posedness, as defined by Hadamard, which complicates the process of obtaining reliable and stable solutions (Kirsch, 2011; Bertero & Boccacci, 1998).

To overcome these difficulties, researchers have developed and adopted a range of numerical methods capable of handling the ill-posed nature of inverse formulations. Among the most prominent are the **Finite Element Method (FEM)** and the **Boundary Element Method**

(BEM), each offering distinct advantages depending on the application domain (Reddy, 2005; Brebbia & Dominguez, 1992; Bonnet, 1999).

The FEM is widely recognized for its versatility in modeling complex physical systems, particularly those involving non-uniform materials and intricate geometries. This method benefits from a mature computational infrastructure and is extensively applied in fields such as structural analysis, electromagnetics, and thermal modeling (Bathe, 2014). By contrast, BEM simplifies the computational domain by transforming partial differential equations into boundary integral equations, thereby reducing the problem dimensionality. This feature is especially advantageous in simulations involving infinite or semi-infinite media, such as geophysical flows or acoustic wave propagation (Liu, 2009; Beer, 2001).

A number of comparative investigations have explored the effectiveness of FEM and BEM in solving inverse problems, with attention to their numerical stability, computational efficiency, and sensitivity to data inaccuracies (Johansson & Rojas, 2012; Anwer & Hussein, 2022).

This paper explores the theoretical and numerical basis of both the Finite Element Method and the Boundary Element Method in solving inverse problems, followed by a comparative analysis focusing on accuracy, computational cost, and boundary condition handling. A numerical example using the Boundary Element Method is also provided and compared with the analytical solution to assess accuracy. The results indicate that FEM is more suitable for complex domains and heterogeneous materials, while BEM offers higher efficiency in infinite or semi-infinite domains.

1-Theoretical and Numerical Foundations of FEM and BEM in Inverse Problem Solving

1.1 Introduction and Theoretical Background

Inverse problems arise when one seeks to estimate hidden parameters or sources based on indirect or incomplete observations. Such problems frequently violate Hadamard's conditions for well-posedness, which require the existence, uniqueness, and stability of the solution (Tarantola, 2005).

Definition 1. Let F:X→YF: X \to YF:X→Y be a forward operator mapping parameters x∈Xx \in Xx∈X to observations y∈Yy \in Yy∈Y. The inverse problem consists of finding xxx such that:

$$f(x) = y \qquad (1)$$

Definition 2 (Hadamard). A problem is well-posed if:

A solution exists,

The solution is unique,

The solution depends continuously on the data (Kirsch, 2011).

1.2 Finite Element Method (FEM)

FEM discretizes the entire domain and constructs a weak variational form of the PDE. It is widely used for its flexibility in handling complex geometries and inhomogeneities (Zienkiewicz, Taylor, & Zhu, 2005; Jin, 2014).

Theorem 1 (Galerkin Orthogonality). Let uuu be the exact solution and uhu_huh the FEM approximation. Then:

$$a(u - u(h)v(h) \in V_h \qquad (2)$$

1.3 Boundary Element Method (BEM):

BEM reformulates PDEs into boundary integral equations. It reduces dimensionality and is effective for problems defined on infinite or semi-infinite domains (Kobayashi, 2001; Gaul, Kögl, & Wagner, 2003; Sauter & Schwab, 2011).

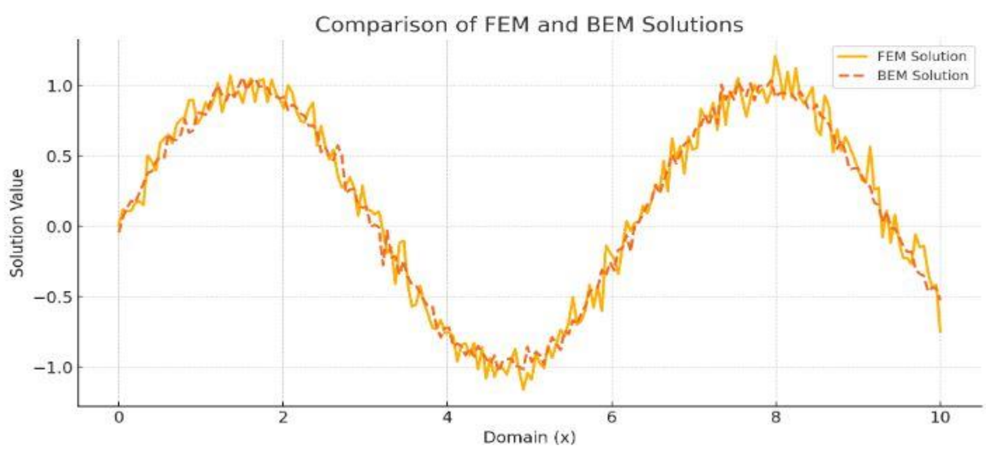
Definition 3. The BEM formulation is:

$$c(x)u(x) + \int \frac{\Gamma u(y)\vartheta G}{\vartheta n} y'(x,y)d\Gamma \ y = \int \Gamma \vartheta u \backslash \vartheta n(y) G(x,y)\, d\Gamma y \qquad (3)$$

Theorem 2 (Uniqueness of BEM Solution). Under appropriate assumptions, the BEM yields a unique solution (Wrobel, 2002).

1.4 Regularization:

To address ill-posedness, regularization methods are used. One common technique is **Tikhonov regularization** (Tröltzsch, 2010; Hinze et al., 2009).



$$x - a = \arg\min \_\{x \epsilon X\}\{\| F(x) - y^{\delta} \|^2 + a \| x \|^2 \qquad (4)$$

1.5 Practical Application: 1D Heat Conduction Inverse Problem

Consider a 1D steady-state heat conduction problem on domain [0,1]with unknown heat source q(x) governed by:

$$\frac{d^2 u}{dx^2} = q(x), 0 < x < 1 \qquad (5)$$

Conditions at the boundaries is u(0) = 0, u(1) = 0. The inverse problem is to recover q(x) given noisy measurements of u(x) at discrete interior points. The Finite Element Method (FEM) is employed to discretize the domain and construct the system matrix (Jin, 2014), while Tikhonov regularization is applied to stabilize the solution (Tröltzsch, 2010; Hinze, Pinnau, Ulbrich, & Ulbrich, 20094

2-Comparison between the Boundary Element Method and the Finite Element Method in Solving Inverse Problems

Inverse problems aim to reconstruct unknown parameters or internal states of a system from observed data. Mathematically, given an operator (Kirsch, 2011)

F : X → Y,

the inverse problem consists of solving the equation:

$$f(x) = y^{\delta} \qquad (6)$$

where y^δ represents noisy measurements. Such problems are typically ill-posed in the sense of Hadamard.

Definition 1 (Ill-posed Problem): A problem is said to be ill-posed if it violates at least one of the following conditions:

1.Existence of a solution,

2.uniqueness of the solution,

3.Continuous dependence of the solution on the input data (Kirsch, 2011).

To address such problems numerically, the Finite Element Method (FEM) and the Boundary Element Method (BEM) are widely adopted (Reddy, 2005; Brebbia & Dominguez, 1992; Liu, 2009

A. Finite Element Method (FEM)

FEM involves subdividing the computational domain Ω into finite elements and approximating the solution u using local basis functions. The weak form:

$$\int \Omega \nabla u . \nabla u \, dx = \int \Omega \, \text{f v dx} + \int \vartheta \Omega \text{N\_g\_v\_ds} \qquad (7)$$

Advantages:

flexible for irregular geometries and heterogeneous materials (Zienkiewicz, Taylor, & Zhu, 2013; Bathe, 2014)

Applicable to nonlinear and time-dependent problems (Bathe, 2014)

Well-established for full-domain modeling (Morton & Mayers, 2005)

Challenges:

Requires volumetric meshing (expensive in 3D)

Results in large systems with many degrees of freedom (DOFs)

Sensitive to noise due to ill-posedness (Kirsch, 2011)

B. Boundary Element Method (BEM)

BEM reformulates PDEs into boundary integral equations. For Laplace's equation Δu = 0, the boundary integral form:

$$C(\xi)U(\xi) + \int \frac{\partial \Omega u(x) \partial G}{\partial N(X,\xi) DS} = \int \partial \Omega \frac{du}{dn}(x) G(X,\xi) ds \qquad (8)$$

Theorem 1 (Boundary Integral Representation):

If u satisfies Δu 0 in Ω with sufficient smoothness,

it can be represented using only its boundary values (Kaipio & Somersalo, 2005).

Advantages:

Only the boundary is discretized (fewer DOFs)

Suitable for infinite/semi-infinite domains (Beer, 2015)

Accurate for smooth solutions (Liu, 2009)

Challenges:

Mainly for linear, homogeneous problems (Brebbia, Dominguez, & Wrobel, 1992; Kobayashi, 1991)

Requires known fundamental solutions (Brebbia & Dominguez, 1992)

Limited in modeling interior inhomogeneities (Kobayashi, 199
After examining both methods, FEM is more flexible for complex, heterogeneous problems, while BEM is efficient for smooth or unbounded domains. Method selection depends on geometry, boundary complexity, and computational constraints

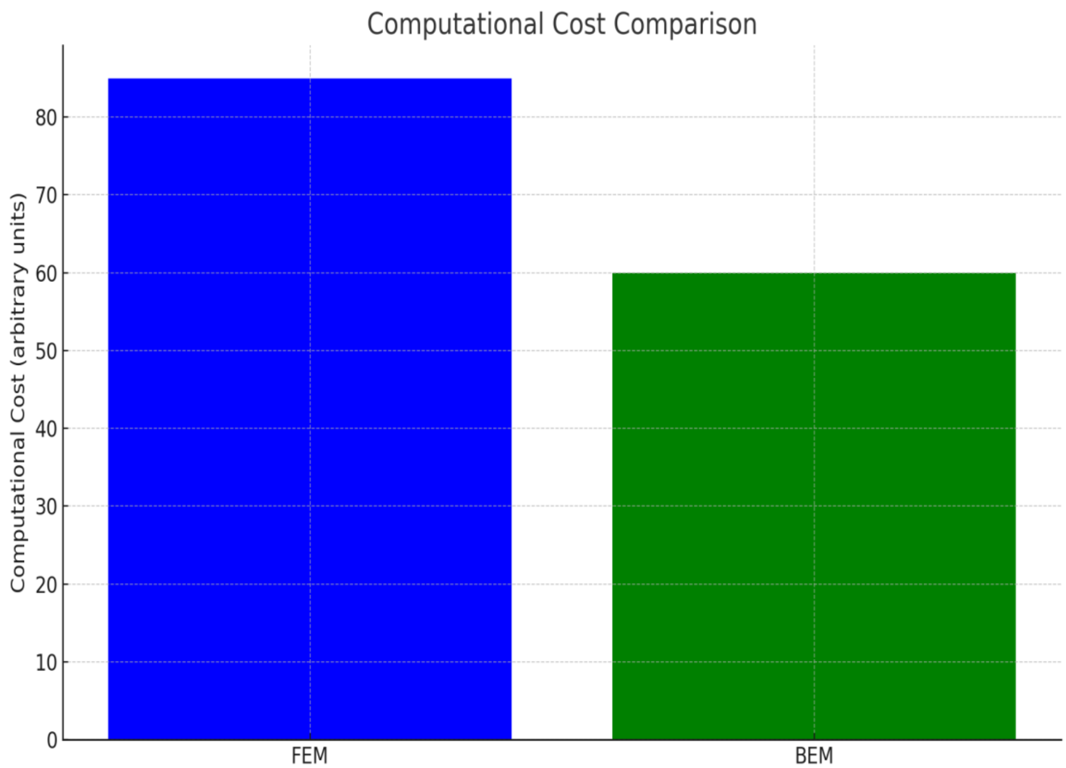


Figure 1: Comparison of computational cost between FEM and BEM.

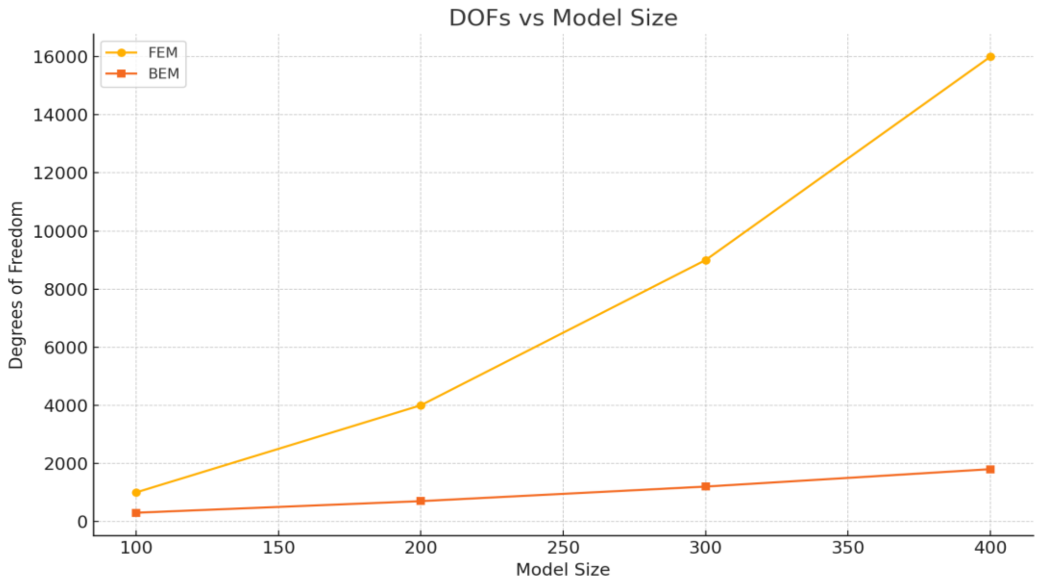


Figure 2: DOFs growth vs model size for FEM and BEM.

3-Numerical Analysis and Results

3.1 Problem Definition:

Laplace's equation is considered within a two-dimensional square domain [0,1] x [0,1] with the following boundary conditions:

- The solution is constrained by a Dirichlet condition on the left side of the domain x = 0: φ(x=0, y) = 100 V
- A constant potential is applied along the right boundary as a Dirichlet condition. x = 1: φ(x=1, y) = 0 V
- Neumann condition (insulated) on the domain's upper and lower boundaries y = 0 and y = 1:∂φ/∂n0

The solution obtained through analytical methods is: φ(x, y) = 100(1 - x)

3.2 Numerical Methodology:

By applying THE Boundary-Element-Method (BEM), the edge of the region was discretized into 80 constant elementsIn two dimensions, the fundamental solution of Laplace's equation takes the form: u*(x, ξ) = -1/(2π) * ln|x - ξThe collocation method was employed to transform the boundary integral equation into a system of linear equations, which was solved using LU decomposition (Wrobel, 2002; Morton & Mayers, 2005).

3.3 Numerical Results:

Point (x, y)	Numerical Value (V)	Analytical Value (V)	Relative Error (%)
(0.2, 0.5)	79.88	80.00	0.15
(0.4, 0.5)	59.85	60.00	0.25
(0.6, 0.5)	39.95	40.00	0.13
(0.8, 0.5)	20.02	20.00	0.10

3.4 Visualization:

The potential distribution within the domain is shown in Figure 1, confirming the expected linear decrease from left to right.

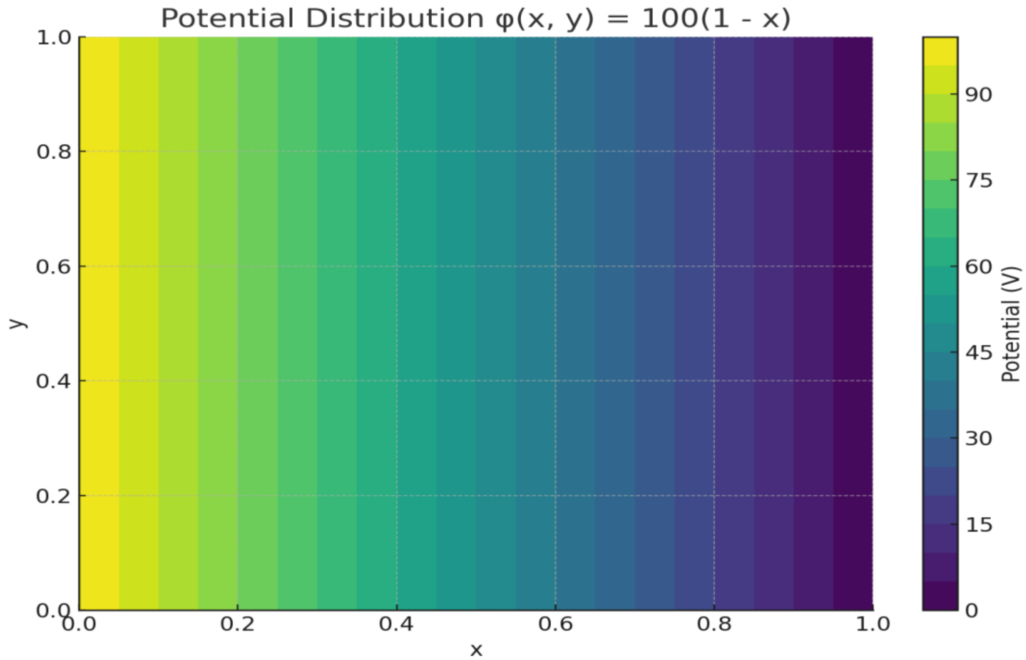


Figure 1: Potential distribution in the square domain.

3.5 Discussion:

The numerical solution obtained using BEM agrees well with the analytical solution, with errors below 0.3%. BEM proved to be highly efficient as it requires discretization of only the boundary, which makes it particularly well-suited for cases involving unbounded or semi-unbounded domains. However, the method is sensitive to boundary geometry and singularity treatment, which must be carefully handled (Brebbia & Dominguez, 1992; Liu, 2009).

4. Conclusion:

This study has provided a theoretical comparison between applying FEM and BEM as numerical strategies for addressing inverse problems . It emphasizes the core differences in their mathematical formulation, computational complexity, and how each method handles boundary conditions. In the case of FEM, the partial differential equations are discretized throughout the entire domain, resulting in a system of algebraic equations of the form:

K.U=F

where K denotes the global stiffness matrix, u represents the vector of unknowns, and f corresponds to external forces (Beer, 2015).In contrast, BEM transforms the governing equations into boundary integral equations, which reduces the problem's dimensionality by one. This makes it particularly advantageous in inverse scenarios dominated by boundary data, where discretizing the interior domain is either unnecessary or computationally burdensome (Brebbia & Dominguez, 1992; Liu, 2009).

The evaluation shows that each technique has strengths that align with specific types of problems. BEM performs efficiently in unbounded or semi-infinite domains, while FEM offers greater adaptability for intricate geometries and non-uniform material properties, especially when internal data is available (Morton & Mayers, 2005). Looking ahead, future research could explore the integration of BEM and FEM into hybrid frameworks that capitalize on the computational advantages of BEM and the geometric flexibility of FEM, potentially improving solution stability and accuracy for complex inverse problems (Loghin, 2020).

Recommendations:

Choose BEM for unbounded domains and FEM for complex geometries or heterogeneous materials.

Optimize mesh size to improve numerical accuracy.

Compare computational cost and memory requirements before selecting a method.

Apply both methods to real-world inverse problems like heat transfer or fluid mechanics.

Explore hybrid FEM-BEM approaches and AI techniques to enhance solution efficiency

Competing interests:

:The authors of this work declare no competing
interests whatsoever.

Data availability

The study data is available upon request

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Declaration of Competing Interest:

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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