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Metric areas and results of best periodic points

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ABSTRACT:

This thesis consists of four parts. In the first chapter, the basic definitions and theorems that will be used throughout the thesis are given. In the second part, we obtain some of the best periodic proximity point results in metric spaces for p-contraction type mappings. Thus, we generalize and improve the similar results existing in the literature. In the third section, we demonstrate some of the best periodic proximity point results in metric spaces for nonunique contraction mappings. Some generalizations of both the fixed point and best proximity point results have been obtained for nonunique contraction mappings.

Keywords: Metric space, Best periodic proximity point, Best proximity point

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1. INTRODUCTION:

Fixed point theory studies have emerged to seek answers to the questions under which conditions a fixed point exists in ordinary differential equations, whether it is unique and how it can be found if it is odd. However, optimization, computational algorithms, physical mathematical modeling, economics, variational, inequalities, complementary problems, balance problems, social sciences, medicine, communication, etc. It shows widespread in many scientific fields. In the field of mathematics, it appears with applications in many fields such as general topology, functional analysis, nonlinear functional analysis, operator theory, differential equations, potential theory, approximation theory, control systems and games.

Fixed point theory deals with equations such as Tx=0 and their solution. Finding a fixed point for a particular operator suitable for the type of equation means finding the solution to the problem. One of the first studies on fixed point theory in metric spaces was a fundamental result in 1922 known as the Banach contraction principle (Banach 1922). The Banach contraction principle not only guarantees the existence of the fixed point of the transformation, but also shows the uniqueness of this fixed point and how it can be found. Then the Banach contraction principle is extended to different areas (De la Sen 2010, Hussain et al. 2017, Jleli and Samet. 2014, Kadelburg and Radenovic 2014, Kirk et al. 2003, Mlaiki et al. 2018, Meir and Keeler 1969, Pant 2014, Proinov 2006, Reich 1972, Singh et al. 2011, Vetro 2015, Yildirim 2018). Popescu (2008) obtained a generalization of this principle by introducing p-contraction mappings. However, in nonlinear systems that are frequently used to solve real-life problems, the solution may not be unique. Therefore, also to consider theorems that do not guarantee the uniqueness of the fixed point is effective in practice. In this sense, Ćirić (1974) proved a nonunique fixed point result. Also, Ćirić (1974) introduced a concept of periodic point. Since every fixed point is a periodic point, periodic point results are a generalization of fixed point results. Therefore, many authors have been studied this topic (Achari 1976, Aslantas and Bachay 2021, Azennar et al. 2019, Carl 1969, Kumari 2013, Onsod et al. 2020, Pachpatte 1979, Senapati et al. 2019)

On the other hand, Basha and Veeramani (1977) presented the notion of the best proximity point. Let Ᵽ,Ʀ be nonempty set in metric space (X,ρ) and T:Ᵽ→Ʀ be a mapping. If Ᵽ∩Ʀ=∅, then there does not exist a point x satisfying Tx=x. Then, it makes sense to consider the existence of an approximate solution x for the minimization problem 〖inf〗_(x∈Ᵽ) ρ(x,Tx). Since ρ(x,Tx)≥ρ(Ᵽ,Ʀ) the most appropriate solution of the mentioned problem is the point x^* that satisfies the condition ρ(x^*,Tx^*)=ρ(Ᵽ,Ʀ). This the point x^* is called the best proximity point of T. This subject has been studied by many authors later on (Altun et al. 2020, Aslantas 2021, Aslantas et al. 2021b, Basha 2010, Basha 2011, Di Bari et al. 2008, Karapınar 2012, Sahin et al. 2020, Vetro 2010) Moreover, Chen and Lin (2012) presented a notion of best periodic proximity point which is a generalization of best proximity point.

In this thesis, we give basic definitions and theorems that will be used throughout the thesis in first chapter. In the second chapter, we give some best periodic proximity point results for p-contraction type mappings on metric spaces. Thus, we generalize and extend similar results. In the third chapter, we prove some best periodic proximity point results for nonunique contraction mappings on metric spaces. Some generalizations of both the best proximity point and fixed point results for nonunique constraction mappings are obtained. In the last section, discussion and recommendation are given.

In this section, some definitions and theorems that we use in other parts of our study are given.

2. Metric Spaces:

Definition 2.1. assume that X be a nonempty set. The function $\rho: \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ is which is called metric (distance function) if it satisfies the following conditions

i) $\rho(x, y) \geq 0$ for all $x, y \in X$.

ii) $\rho(x, y) = 0$ if and only if $x = y$.

iii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y \in X$.

Also, (X, ρ) which is called metric space.

Now, we give some examples for explain this definition.

Remark 2.3. In metric space (X, ρ), the distance between two nonempty subset Ᵽ and Ʀ is denoted by

$$\rho(Ᵽ, Ʀ) = \inf\{\rho(x, y): x \in Ᵽ, y \in Ʀ\}$$

Definition 2.4. the metric space (X, ρ) which is called be bounded metric space if there exist positive number r where $\rho(x, y) \leq r$ for each $x, y \in X$.

Definition 2.5. Assume that (X, ρ) be a metric space, then open ball in X has radius $r > 0$ centered x_0 as follow by

$$B_r(x_0) = \{x \in X: \rho(x, x_0) < r\}$$

and the closed ball in X is defined by

$$\overline{B_r}(x_0) = \{x \in X: \rho(x, x_0) \leq r\}$$

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Definition 2.6. Let (X, ρ) and $(Y, \rho^{\backslash})$ be two different metric spaces with $Ᵽ \subseteq X$. A function $f: Ᵽ \to Y$ which is called continuous at $a \in Ᵽ$, if for each $\varepsilon > 0$ there is $\delta > 0$ where if $\rho(x, a) < \delta$ then $\rho^{\backslash}(f(x), f(a)) < \varepsilon$. If f is continuous at all $a \in Ᵽ$, then f is continuous on Ᵽ.
We gives the definitions of the convergent sequence and Cauchy sequence in metric space.

Definition 2.7. A sequence $\{x_n\}$ in metric space (X, ρ) which is called convergent to x if for all $\varepsilon > 0$, if there is positive integer N where
$\rho(x_n, x) \leq \varepsilon$ for each $n \geq N$.
and said to be Cauchy sequence if
$\rho(x_n, x_m) \leq \varepsilon$ for each $n, m \geq N$.

We will know that every convergent sequence is Cauchy sequence, but the converse is not necessary to be true. To explain that consider the usual metric space $(\mathbb{R} - \{0\}, \rho)$, then the sequence $\left\{\frac{1}{n}\right\}$ is Cauchy but not convergent in $(\mathbb{R} - \{0\}, \rho)$.

Definition 2.8. If for every Cauchy sequenc in metric space (X, ρ) is convergent sequence, then (X, ρ) is said to be complete metric space.

Theorem 2.9. Let (X, ρ) and $(Y, \rho^{\backslash})$ be two different metric spaces. Then the function $f: X \to Y$ is said to continuous at $b \in X$, if and only if $f(x_n) \to f(b)$ in Y, whenever $x_n \to b$ in X, for every sequence $\{x_n\}$ in X.

Some Properties Of Fixed Points and Periodic Points:

In this section, we give some fundamental definitions and results about fixed point of the mapping T known on a metric space X, with necessary condition for exists and unique of fixed point

Definition 2.10. the mapping T define on a metric space (X, ρ) which is called has a fixed point $x \in X$ if satisfy $Tx = x$.

To illustrate this definition, consider the following example

Example 2.11. Consider the mapping $T: \mathbb{R} \to [0,1]$ is defined by $Tx = \sin x$ its clear that 0 is fixed point of T but $(\pi/2)$ can not be fixed point of T.

Remarks 2.12. i) Not necessary all mappings defined on metric spaces has fixed point, see the mapping $T: (X, \rho) \to (X, \rho)$ follow as $Tx = x + a$, where $a \neq 0$ is a real number not has fixed point.
ii) Some types of mappings defined on metric space has only one fixed point such as $T: \mathbb{R} \to [0,1]$ defined by $Tx = \sin x$, and the fixed point of this mapping is only zero.
iii) The identity mapping on a metric has infinite set of fixed points.

iv) If x is a fixed point of $T: (X, \rho) \to (X, \rho)$, then x is not necessary by fixed point of $T|_{Ᵽ}: Ᵽ \to X$, where $Ᵽ \subseteq X$. To illustrate that see, $T: \mathbb{R} \to \mathbb{R}$ defined by $Tx = \frac{x}{2}$ has a fixed point at $x = 0$, but $T|_{(0,1)}: (0,1) \to X$, has no fixed point.

By using Hine-Boreal theorem, one can have the following corollary:

Corollary 2.29. assume that $T: X \to X$ be a continuous mapping define on same space on T-orbitally complete metric space(X, ρ). If T satisfies the follow conditions

$$\min\{\rho(Tx, Ty)\rho(x, y), \rho(x, Tx), \rho(y, Ty)\} - a \min\{\rho(x, Ty), \rho(y, Tx)\} \leq q \max\{\rho(x, y), \rho(x, Tx),$$

where $a \geq 0$ and $0 \leq q < 1$, then T has a fixed point.

Definition 2.30. Let $X \neq \emptyset$ with $T: X \to X$ be a mapping, if there is $n \in N$ where $T^n a = a$, then a is said to be period point of T.

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Theorem 2.31. assume that $T: X \to X$ be an orbitally continuous mapping of a T- orbitally complete metric space X into itself and $\varepsilon > 0$. If there is a point a_0 in X where $\rho(a_0, T^k a_0) < \varepsilon$ for some $k \in I$ and if T satisfies $0 < \rho(a,b) < \varepsilon$ implies

$$\min\{\rho(Ta,Tb), \rho(a,Ta), \rho(b,Tb)\} \leq q\rho(a,b)$$

for some $q < 1$ and each $a,b \in X$. Then, T has a periodic point.

The Best Proximity Points and Best Periodic Proximity Points

In this section, we give some properties of best proximity point and best periodic proximity points of mappings defined on metric space with illustrate these concepts.

Definition 2.32. assume that (X,ρ) be a metric space and Ᵽ and Ʀ are non-empty sets of X, a mapping T: Ᵽ → Ʀ has a best proximity x in A if $\rho(x,Tx) = \rho($Ᵽ, Ʀ$)$.

In fact that we study the proximity point when the a non-self mapping T: Ᵽ → Ʀ does not necessarily have a fixed point, it mean that the equation $Tx = x$ has no solution.

Theorem 2.33. Let (X,ρ) be a complete metric space, Ᵽ and Ʀ be non-empty sets of X, where Ᵽ and Ʀ are closed and T : Ᵽ∪Ʀ → Ᵽ∪Ʀ be a mapping. Assume that T is a cyclic mapping, that is, T(Ᵽ) ⊆ Ʀ anρ T(Ʀ) ⊆ Ᵽ. If it exists k in [0,1) such that

$$\rho(Tx,Ty) \leq k\rho(x,y) \qquad (2.1)$$

for all x ∈ Ᵽ and y ∈ Ʀ, then T has a fixed point in Ᵽ ∩ Ʀ.

Note that, unlike Banach contraction principle, the mapping Λ is not necessary to be continuous in Theorem 2.32. Because of its applicability, there are many studied on this subject in the literature. In this sense, Eldred and Veeramani (2006) introduced a concept of cyclic contraction mapping by considering $A \cap B = \emptyset$ and proved a fundamental result.

Definition 2.34. assume that (X,ρ) be a metric space, Ᵽ and Ʀ be non-empty sets of X. Then, a mapping T: Ᵽ∪Ʀ → Ᵽ∪Ʀ is called a cyclic contraction if it satisfies T(Ᵽ) ⊆ Ʀ and T(Ʀ) ⊆ Ᵽ and the following condition

$\rho(Tx,Ty) \leq k\rho(x,y) + (1-k)\rho($Ᵽ, Ʀ$)$

for all x ∈ Ᵽ and y ∈ Ʀ, where $k \in [0,1)$.

Taking into account this situation, the Inequality (2.1) in Theorem 2.33 has been extended different way from the results. Consequently, they presented the following result.

Theorem 2.35. Let (X,ρ) be a metric space, Ᵽ and Ʀ be nonempty subsets of X and f: Ᵽ∪Ʀ → Ᵽ∪Ʀ be a cyclic contraction mapping. Let x_0 ∈ Ᵽ and follow as $x_{n+1} = Tx_n$, for all $n \geq 1$. If $\{x_{2n-1}\}$ has a subsequence convergent in Ᵽ, then it exists $x \in A$ where $\rho(x,Tx) = \rho($Ᵽ, Ʀ$)$.

Consequently, they gave a generalization of Theorem (2.33). Indeed, if $A \cap B$ is empty, then T cannot have a fixed point. In this case, it is sensible to find the existence of solution of the minimization problem $\min\{\rho(x,Tx) :$ x ∈ Ᵽ$\}$. Since $\rho(x,Tx) \geq \rho($Ᵽ, Ʀ$)$ for all x ∈ Ᵽ, a point x satisfying $\rho(x,Tx) = \rho($Ᵽ, Ʀ$)$ is an optimal solution of the minimization problems $\min_{x \in Ᵽ} \rho(x,Tx)$. This point x is called a best proximity point of T. As, in special case Ᵽ = Ʀ = X, every best proximity point becomes a fixed point, there are many works on this topic. (Altun et al. 2020, Basha 2011, Sahin et al. 2020)

Let (X,ρ) which is a metric space, Ᵽ, Ʀ be non-empty sets of X and T: Ᵽ → Ʀ be a mapping. We regard the subsets of Ᵽ and Ʀ, respectively.

$$Ᵽ_0 = \{x \in Ᵽ : \rho(x,y) = \rho(Ᵽ, Ʀ) \text{for some } x \in Ʀ\}$$

and

$$Ʀ_0 = \{x \in Ʀ : \rho(x,y) = \rho(Ᵽ, Ʀ) \text{for some } x \in Ᵽ\}$$

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where $\rho(\text{Ᵽ}, \text{Ʀ}) = \inf\{\rho(x,y): x \in \text{Ᵽ} \text{ an}\rho\ y \in \text{Ʀ}\}$. Basha (2011) proved a version of the Banach contraction principle for nonself mappings by introducing the concept of proximal contraction mapping.

Now, the following theorem gives grantees conditions to get the unique proximity point for the contraction cyclic map.

1- **Definition 3. 1.** Let (X, ρ) be a metric space and Ᵽ and Ʀ are nonempty subset of X, a cyclic mapping $T: \text{Ᵽ} \cup \text{Ʀ} \to \text{Ᵽ} \cup \text{Ʀ}$ which is said to be cyclic weaker Meir-Keeler contraction mapping if satisfies the following conditions

i) $T(\text{Ᵽ}) \subset \text{Ʀ}$ and $T(\text{Ʀ}) \subset \text{Ᵽ}$.

ii) There is $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ is a ϕ −mapping in X, such that for each $n \in N$ and $x \in \text{Ᵽ}, y \in \text{Ʀ}$ with $\rho(x,y) - \rho(\text{Ᵽ}, \text{Ʀ}) > 0$, $\rho(f^n x, f^n y) - \rho(\text{Ᵽ}, \text{Ʀ}) < \phi^n(\rho(x,y) - \rho(\text{Ᵽ}, \text{Ʀ})$, and if $\rho(x,y) - \rho(\text{Ᵽ}, \text{Ʀ}) = 0$ implies to $\rho(f^n x, f^n y) - \rho(\text{Ᵽ}, \text{Ʀ}) = 0$.

We recall the definition of best periodic proximity point

Definition 3. 2. Let Ᵽ, Ʀ be nonempty subsets of a metric space(X, ρ) and $T: \text{Ᵽ} \cup \text{Ʀ} \to \text{Ᵽ} \cup \text{Ʀ}$ be a cyclic mapping. A point $z \in \text{Ᵽ} \cup \text{Ʀ}$ whch is called a best periodic proximity point if $\rho(u, T^m u) = \rho(\text{Ᵽ}, \text{Ʀ})$ for some $m \in \mathbb{N}$.

A point $x \in \text{Ᵽ}$ is called a periodic point of a self mapping Tif there exists $m \in \mathbb{N}$ satisfying $Tmx = x$. It is clear that if we take $\text{Ᵽ} = \text{Ʀ} = X$ in Definition (2.42), it becomes the definition of periodic point of T.

Now, the following theorem gives the conditions for get the best periodic proximity point of.

Theorem 3. 3. assume that (X, ρ)be a metric space, Ᵽ and Ʀ be non-empty sets of X. with $T: \text{Ᵽ} \cup \text{Ʀ} \to \text{Ᵽ} \cup \text{Ʀ}$ is a cyclic weaker Meir-Keeler contraction. If for $x_0 \in \text{Ᵽ}$, the sequence $\{T^{2n-1}x\}$ converges to $x_0 \in \text{Ᵽ}$, then x is said to be a best periodic proximity point of T in Ᵽ.

4- SOME RESULTS FOR NONUNIQUE CONTRACTION MAPPINGS:

We begin to this section by giving the following result.

Theorem 4.1. Let A, B be nonempty subsets of a metric space (X, ρ) and $T: A \cup B \to A \cup B$ be an orbitally continuous cyclic mapping. Define$x_{n+1} = Tx_n$for alln $\in \mathbb{N}$. Assume that there is a convergent subsequence $\{x_{2n_k}\}_{k\in\mathbb{N}}$ of $\{x_{2n}\}$ in Ᵽ. If there exists $x_0 \in A$such that $\rho(x_0, x_{2t-1}) < \rho(A,B) + \varepsilon$for some $t \in \mathbb{N}$and Tsatisfies

$$0 < \rho(x,y) < \rho(A,B) + \varepsilon$$

implies

$$\min\{\rho(Tx,Ty), \rho(x,Tx), \rho(y,Ty)\} \leq s\rho(x,y) + (1-s)\rho(A,B) \tag{4.1}$$

for some $s \in [0,1)$ and for all $x \in A, y \in B$, then T has a best periodic proximity point.

Proof. Let$K = \{t: \rho(x_0, x_{2t-1}) < \rho(A,B) + \varepsilon$ for some $x_0 \in A\}$. Becauseof the our assumption, K is nonempty set. Let $\min K = m$ and $x_0 \in A$ such that $\rho(x_0, x_{2m-1}) < \rho(A,B) + \varepsilon$. Assume that $m = 1$, that is, $\rho(x_0, x_1) < \rho(A,B) + \varepsilon$. Taking $x = x_0 \in A$ and $y = x_1 \in B$ in Equation (4.1), we have

$$\min\{\rho(Tx_0, Tx_1), (\rho x_0, Tx_0), \rho(x_1, Tx_1)\} \leq s\rho(x_0,x_1) + (1-s)\rho(A,B)$$

which implies

$$\min\{\rho(x_0,x_1), \rho(x_1,x_2)\} \leq s\rho(x_0,x_1) + (1-s)\rho(A,B) \tag{4.2}$$

If $\min\{\rho(x_0,x_1), \rho(x_1,x_2)\} = \rho(x_0,x_1)$,then it is easy that x_0 is a best proximity point and so the proof is completed. Otherwise, from Equation (4.2), we have

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$$\rho(x_1,x_2) \le s\rho(x_0,x_1) + (1-s)\rho(A,B)$$
$$< s(\rho(A,B)+\varepsilon) + (1-s)\rho(A,B)$$
$$= s\,\varepsilon + \rho(A,B) \qquad (4.3)$$

From Equation (4.3), we get

$$\rho(x_1,x_2) \;\; < s\varepsilon + \rho(A,B)$$
$$< \rho(A,B) + \varepsilon$$

Again taking $x = x_0 \in A$and $y = x_1 \in B$in (4.1), we have

$$\min\{\rho(x_1,x_2),\rho(x_2,x_3)\} \le s\rho(x_1,x_2) + (1-s)\rho(A,B)$$

If $\min\{\rho(x_1,x_2),\rho(x_2,x_3)\} = \rho(x_1,x_2)$ then it is easy that x_1 is a best proximity point the proof is completed. Otherwise, from Equation (4.2), we have

$$\rho(x_2,x_3) \le s\rho(x_1,x_2) + (1-s)\rho(A,B)$$
$$< s\big(s\,\varepsilon + \rho(A,B)\big) + (1-s)\rho(A,B)$$
$$= s^2\varepsilon + \rho(A,B) < \rho(A,B) + \varepsilon$$

Continuing this process, we can construct a sequence$\{x_n\}$such that$x_{n+1} = Tx_n$for all$n \in \mathbb{N}$ and

$$\rho(x_{2n-1},x_{2n}) < \rho(A,B) + \varepsilon$$

for all $n \in \mathbb{N}$,. Then, we have, from Equation (4.1),

$$\min\{\rho(Tx_{2n-1},Tx_{2n}),\rho(x_{2n-1},Tx_{2n-1}),\rho(x_{2n},Tx_{2n})\} \le s\rho(x_{2n-1},x2n) + (1-s)\rho(A,B)$$

which implies that

$$\min\{\rho(x_{2n},x_{2n+1}),\rho(x_{2n-1},x_{2n})\} \le s\rho(x_{2n-1},x_{2n}) + (1-s)\rho(A,B).$$

If there exists$n0 \in N$ such that$\rho\big(x_{2n_0-1},x_{2n_0}\big) < \rho\big(x_{2n_0},x_{2n_0+1}\big)$, then x_{2n_0-1} is a best proximity point the proofis completed.

Hence, assume $\rho(x_{2n},x_{2n+1}) \le \rho(x_{2n-1},x_{2n})$ for all $n \in \mathbb{N}$. Then, we have

$$\rho(x_{2n},x_{2n+1}) \le s\rho(x_{2n-1},x_{2n}) + (1-s)\rho(A,B)$$

for all $n \in \mathbb{N}$. Therefore

$$\rho(x_{2n+1},x_{2n+2}) \le s\rho(x_{2n},x_{2n+1}) + (1-s)\rho(A,B)$$
$$\le s(s\rho(x2n-1,x2n) + (1-s)\rho(A,B)) + (1-s)\rho(A,B)$$
$$= s^2\rho(x_{2n-1},x_{2n}) + (1+s)(1-s)\rho(A,B)\vdots$$
$$\le\; s^{2n}\rho(x_0,x_1) + (1+s+s2+\cdots+s^{2n-1})(1-s)\rho(A,B)$$
$$= s2n\rho(x0,x1) + \frac{(1-s2n)}{1-s}\,\rho(A,B)$$

for all $n \in \mathbb{N}$. Therefore, we get

$$\rho(x_{2n+1},x_{2n+2}) \to \rho(A,B) \text{ as } n \to \infty \qquad (4.4)$$

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We can also show that $\rho(x_{2n}, x_{2n+1}) \to \rho(A, B)$ xs $n \to \infty$. Assume that $x_{2n_k} \to x^*$ for some $x^* \in A$. Since T is an orbitally continuous mapping, we have

$$x_{2n_k+1} = Tx_{2n_k} \to Tx^* \text{ as } n \to \infty$$

From Equation (4.4), we have $\rho(x^*, Tx^*) = \rho(A, B)$ and so T has a best proximity point x^* in A which is a best periodic proximity point. Now, assume $m \geq 2$. That is,

$$\rho(x, Tx) \geq \rho(A, B) + \varepsilon \qquad (4.5)$$

for all $x \in A$. Then, since $\rho(x_0, x_{2m-1}) < \rho(A, B) + \varepsilon$. Replacing $x = x_0$ and $y = x_{2m-1}$ in Equation (4.1),

$$\min\{\rho(Tx_0, Tx_{2m-1}), \rho(x_0, Tx_0), \rho(x_{2m-1}, Tx_{2m-1})\} \leq s\rho(x0, x_{2m-1}) + (1-s)\rho(A, B)$$

which implies that

$$\min\{\rho(x_1, x_{2m}), \rho(x_0, x_1), \rho(x_{2m-1}, x_{2m})\} \leq s\rho(x0, x_{2m-1}) + (1-s)\rho(A, B)$$

From Equation (4.5) we get

$$\begin{aligned} \rho(x_1, x_{2m}) &\leq s\rho(x0, x_{2m-1}) + (1-s)\rho(A, B) \\ &< s(\rho(A, B) + \varepsilon) + (1-s)\rho(A, B) \\ &= s\varepsilon + \rho(A, B) \\ &< \rho(A, B) + \varepsilon \end{aligned}$$

Again, taking $x = x_1$ anρ $y = x_{2m}$ in Equation (4.1)

$$\min\{\rho(Tx_1, Tx_{2m}), \rho(x_1, Tx_1), \rho(x_{2m}, Tx_{2m})\} \leq s\rho(x_1, x_{2m}) + (1-s)\rho(A, B)$$

which implies that

$$\min\{\rho(x_2, x_{2m+1}), \rho(x_1, x_2), \rho(x_{2m}, x_{2m+1})\} \leq s\rho(x_1, x_{2m}) + (1-s)\rho(A, B).$$

Similarly, we have

$$\begin{aligned} \rho(x_2, x_{2m+1}) &\leq s\rho(x_1, x_{2m}) + (1-s)\rho(A, B) \\ &\leq s\big(s\rho(x_0, x_{2m-1}) + (1-s)\rho(A, B)\big) + (1-s)\rho(A, B) \\ &= s^2\rho(x_0, x_{2m-1}) + (1+s)(1-s)\rho(A, B) \end{aligned}$$

Also, since $\rho(x_1, x_{2m}) < \rho(A, B) + \varepsilon$, we get

$$\begin{aligned} \rho(x_2, x_{2m+1}) &\leq s\rho(x_1, x_{2m}) + (1-s)\rho(A, B) \\ &< s(\rho(Ᵽ, Ʀ) + \varepsilon) + (1-s)\rho(A, B) \\ &= \rho(A, B) + s\varepsilon \\ &< \rho(A, B) + \varepsilon. \end{aligned}$$

Continuing this process, we can construct a sequence $\{x_n\}$ such that $x_{n+1} = x_n$ for all $n \in \mathbb{N}$ and

$$\rho(x_{2n}, x_{2n+2m-1}) \leq \rho(A, B) + \varepsilon$$

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for all $n \in \mathbb{N}$. Hence, we have

$$\rho(x_{2n}, x_{2n+2m-1}) \leq s\rho(x_{2n-1}, x_{2n+2m-2}) + (1-s)\rho(A,B)$$
$$\leq s^2\rho(x_{2n-2}, x_{2n+2m-3}) + (1+s)(1-s)\rho(A,B)$$
$$\leq s^{2n}\rho(x_0, x_{2m-1}) + (1-s)\frac{(1-s^{2n})}{1-s}\rho(A,B).$$

we have

$$\rho(x_{2n}, x_{2n+2m-1}) \to \rho(A,B) \ as \ n \to \infty \tag{4.6}$$

Now, assume $x_{2n_k} \to x^*$ as $n \to \infty$ for some $x^* \in Ᵽ$ where $\{x_{2n_k}\}$ is a convergent subsequence$\{x_{2n}\}$. Since T is an orbitally continuous mapping, T^{2m-1} is an orbitally continuous mapping. Then, we have

$$x_{2n_k+2m-1} = T^{2m-1}x_{2n_k} \to T^{2m-1}x^* \ as \ n \to \infty$$

Hence, from Equation (4.6), we have $\rho(x^*, T^u x^*) = \rho(A,B)$, where $u = 2m-1 \in \mathbb{N}^+$. The proof is complete

Since every sequence has a convergent subsequence in compact set, we give the following result.

Corollary 4.7 Let Ᵽ, Ʀ be nonempty subsets of a metric space (X,ρ) where Ᵽ is compact subset, and$T: A \cup B \to A \cup B$ be an orbitally continuous cyclic mapping.If there exists$x_0 \in$ Ᵽsuch that$\rho(x_0, x_{2t-1}) < \rho(A,B) + \varepsilon$for some $t \in \mathbb{N}$ and T satisfies

$$0 < \rho(x,y) < \rho(A,B) + \varepsilon$$

implies

$$\min\{\rho(Tx,Ty), \rho(x,Tx), \rho(y,Ty)\} \leq s\rho(x,y) + (1-s)\rho(A,B)$$

for some $s \in [0,1)$ and for all $x \in A, y \in B$, then T has a best periodic proximity point.

Taking $A = B = X$ in Theorem (4.1) and Corollary (4.2), we obtain the following two results.

Corollary 4.8. Let (X,ρ) be a metric space, and $T: X \to X$ be an orbitally continuous mapping. Define $x_{n+1} = Tx_n$for alln $\in \mathbb{N}$. Assume that there is a convergent subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ in X. If there exists $x_0 \in A$ such that $\rho(x_0, x_{2t-1}) < \varepsilon$ for some $t \in \mathbb{N}$ and T satisfies

$$0 < \rho(x,y) < \varepsilon$$

implies

$$\min\{\rho(Tx,Ty), \rho(x,Tx), \rho(y,Ty)\} \leq s\rho(x,y)$$

for some $s \in [0,1)$and for all $x, y \in X$, then T has a periodic point.

Corollary 4.9. Let(X,ρ) be a compact metric space, and $T: X \to X$ be an orbitally continuous mapping. Define $x_{n+1} = Tx_n$for alln $\in \mathbb{N}$. If there exists $x_0 \in A$ such that $\rho(x_0, x_{2t-1}) < \varepsilon$ for some $t \in \mathbb{N}$ and T satisfies

$$0 < \rho(x,y) < \varepsilon$$

implies

$$\min\{\rho(Tx,Ty), \rho(x,Tx), \rho(y,Ty)\} \leq s\rho(x,y)$$

for some $s \in [0,1)$and for all $x, y \in X$, then T has a periodic point.

CONCLUSIONS AND RECOMMENDATION:

In this thesis, some original best periodic proximity point results are obtained on metric spaces by using the information appeared in some lictuers. First conclution we obtain best periodic proximity points results for some mappings which are p-contraction mappings defined on

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sutiable space . Secondly, we have prove some best periodic proximity point results for nonunique contraction mappings. Some generalizations of both the best proximity point and fixed pointresults for such mappingshave been obtained.

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