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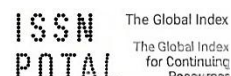
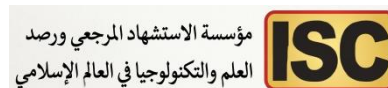
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## The Error Analysis two-dimension nonlinear convection diffusion problem by using Crank-Nicolson Petrov Weak Galerkin

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### Abstract:

In this paper, focuses on the formulation and investigation of two dimensional non-linear Convection diffusion equations using the Crank–Nicolson Petrov weak Galerkin finite element approach. We apply weak Galerkin finite element method and Petrov Galerkin finite element mothded. we defining a space of the Petrov Weak Galerkin finite element method that depends on a test function space differs from the trial function space the value a constant stability parameter of the delta parameter. The stability and error estimate full-discrete is proved and the approximate solutions are convergent with the error of  $O(h + \tau)$ . The analysis addresses stability properties, derives error estimates, and establishes optimal convergence rates in both the  $L^2$  and  $H^1$  norms for the Petrov weak Galerkin finite element PWGFE scheme. In addition, the numerical results of PWGFE method show the convergence and accuracy of the method. several numerical experiments are conducted to validate the theoretical findings.

**Keywords:** Petrov Weak Galerkin Finite Element, Weak Galerkin Finite Element, Nonlinear Convection Diffusion equation, the Stability, the error analysis.

### 1.Introduction:

In this paper, the study of convection–diffusion equations has attracted increasing attention from researchers in engineering and applied mathematics. One of the main difficulties in these problems is the accurate resolution of boundary layer behavior, particularly in regimes, where convective effects significantly outweigh diffusive ones. To address these numerical challenges, stabilization techniques, such as upwind schemes and artificial viscosity are frequently incorporated into finite difference and finite element formulations. In this context, we investigate a Dirichlet boundary value problem for a nonlinear convection–diffusion equation, seeking to determine an unknown function  $u = u(x, t)$

that satisfies the prescribed conditions (Kashkool, H. A)

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(u) \nabla u) + b(u) \cdot \nabla u + cu = f, \quad \Omega \times (0, T], \quad (1.1)$$

$$u(x, t) = g, \quad \Gamma \times (0, T], \quad (1.2)$$

$$u(x, 0) = u^0(x); \quad x \in \bar{\Omega}, \quad (1.3)$$

where  $\Omega \subset \mathbb{R}^2$  with the boundary  $\Gamma$ ,  $\frac{\partial u}{\partial t}$ , and  $x = (x_1, x_2)$ ,  $f = f(x, t) \in \Omega \times (0, T]$  are given functions; also,  $a(u)$  is a diffusion coefficient; and convection–diffusion and nonlinear function are denoted by  $b(u)$ , and  $c = c(x)$  is scalar function on  $\Omega$ . Suppose that the matrix  $a(u)$  satisfies the following property: there exists a constant  $\lambda > 0$  such that

$$\lambda \zeta^T \zeta \leq \zeta^T a(u) \zeta, \forall \zeta \in \mathbb{R}^2.$$

where  $\zeta^T$  is the transpose of vector  $\zeta$ .

Numerical simulations and analysis for the problem (1.1-1.3) have been extensively investigated in recent decades. A wide range of numerical techniques has been developed to address nonlinear convection–diffusion equations. For instance, (Liu and Liu, 2018) investigated numerical approaches for identifying diffusion parameters in nonlinear settings. (Chan *et al.*, 2014) proposed a dual Petrov–Galerkin finite element formulation tailored for higher-order convection–diffusion problems. An upwind finite volume element strategy for nonlinear cases was explored by (Gao and Yuan., 2009), while (Fatehi *et al.*, 2009) extended a modified finite-volume Eulerian–Lagrangian localized adjoint approach to nonlinear convection–diffusion equations. Further contributions include the work of (Havle *et al.*, 2010), who analyzed discontinuous Galerkin finite element methods for the spatial semi discretization of nonlinear, time-dependent convection–diffusion problems, and (Dolejši *et al.*, 2005), who applied similar techniques to nonlinear convection–diffusion models. (Cockburn and Shu, 1998) examined local discontinuous Galerkin methods for nonlinear, time-dependent convection–diffusion systems. (Xikui and Wenhua, 1999) developed a finite element framework for solving transient, multidimensional convection–diffusion equations, whereas (Salhi *et al.*, 2021) employed the Galerkin–characteristic finite element method for time-dependent convection–diffusion problems. The weak Galerkin finite element method has gained considerable attention since its initial introduction by (Wang and Ye, 2013) for general elliptic equations. Subsequently, this method has been widely applied to both linear and nonlinear convection–diffusion problems. (Li and Wang 2013) extended the weak Galerkin framework to parabolic equations, while (Gao *et al.*, 2013) designed numerical schemes based on weak Galerkin methods for certain Sobolev-type problems. Applications to fluid dynamics include the work of (Zhang and Lin, 2019) on stationary Navier–Stokes equations. (Cheichan, and Kashkool, 2019) utilized the weak Galerkin method for one- and two-dimensional nonlinear convection–diffusion equations, as well as for two-dimensional linear problems. (Hussein, 2020) applied this approach to coupled one- and two-dimensional Burger’s equations. Additionally, (Song *et al.*, 2021) proposed an over-penalized weak Galerkin method for elliptic interface problems with discontinuous coefficients and non-homogeneous boundary conditions. (Abdullah and Kashkool, 2022) employed a weak Galerkin mixed finite element formulation to model two-

dimensional incompressible immiscible fluid displacement in porous media. (Abdullah and Kashkool., 2023) introduced the Weak Galerkin Method's Error Convergence for the Two-Dimensional Non-Linear Convection-Diffusion Problem. (Aziz and Hussein., 2022) studied method for the nonlinear Schrodinger problem using weak Galerkin finite elements.

In this study, we focus on a comprehensive analysis of a Petrov weak Galerkin finite element method combined with the Crank–Nicolson time discretization for nonlinear convection–diffusion equations. The structure of the paper is as follows. Section 2 Preliminary. Section 3 introduces the Petrov Galerkin Finite element method. Section 4 introduces the Petrov weak Galerkin Spaces. Section 5 introduces the weak Galerkin variational formulation of the nonlinear convection–diffusion problem. introduces the Petrov weak Galerkin method and Stability properties of the proposed method are established in Section 6. Error estimates, together with convergence results and optimal rates in the  $H^1$ -norm and  $L^2$ -norm under nonlinear assumptions, are derived in Section 7. Finally, Section 8 presents numerical experiments that support and confirm the theoretical findings.

**2.Preliminary:** In this section, we concentrate on the preliminaries, including some definitions

**Definition 2.1: (Weak function)** (wang J. and Ye, X)

Let  $K$  be any polygonal domain with interior  $K^0$  and boundary  $\partial T$ . A weak function on the region  $K$  refers to a function  $v = \{v^0, v^b\}$  such that  $v^0 \in L^2(K)$  and  $v^b \in H^{1/2}(\partial K)$ . The first component  $v^0$  can be understood as the value of  $v$  in the interior of  $K$ , and the second component  $v^b$  is the value of  $v$  on the boundary of  $K$ .

The space of weak function denote by  $W(K)$  is defined as :

$$W(K) = \{v = \{v^0, v^b\}: v^0 \in L^2(K), v^b \in H^{1/2}(\partial K)\}$$

**Definition 2.2:(weak gradient operator)** (wang J. and Ye, X)

For any  $v \in W(k)$ , the weak gradient operator of  $v$  can be defined as a linear functional  $\nabla dv$  in the dual space of  $H(\text{div}, k)$  whose action on each  $q \in H(\text{div}, k)$  is given by

$$(\nabla dv, q) = - \int_K v^0 \nabla \cdot q dk + \int_{\partial K} v^b q \cdot n ds,$$

where  $n$  is the outward normal direction to  $\partial k$ .

**Definition 2.3 :(discrete weak gradient operator)** (wang J. and Ye, X)

Denote by  $Pr(k)$  the set of polynomials on  $k$  with degree no more than  $r$ . For any non-negative integer  $r \geq 0$ , let  $V(K, r) \subset [Pr(K)]^2$  be a subspace of the space of vector-valued polynomials of degree  $r$ . A discrete weak gradient operator, denoted by  $\nabla_{d,r}$ , is defined so that  $\nabla_{d,r} v \in V(K, r) \subset H(\text{div}, k)$  is the unique solution of the following equation

$$\int_K \nabla_{d,r} v \cdot q dk = - \int_K v^0 \nabla \cdot q dk + \int_{\partial K} v^b q \cdot n ds, \forall q \in V(K, r)$$

### 3. Petrov Galerkin Finite Element method: (Keshaish and Kashkool, 2020)

Petrov Galerkin finite element method, whose ansatz(trail) function space and test function space are different. Let  $\psi$  be the ansatz (trail) space and  $\phi$  be the test space with  $\dim \psi = \dim \phi$ , then the Petrov –Galerkin finite element method reads as follows: Find  $u \in \psi$  such that

$$a_{PG}(u, v) = f(v) \quad \forall v \in \phi.$$

Where

$$\psi = H_0^1(\Omega) = \{v: v \in H^1(\Omega); v|_T = 0\}$$

$$\phi = \{w: w = v + \delta \beta \cdot \nabla v; v \in \psi\}$$

**4. Petrov Weak Galerkin Spaces:** When a test function space differs from the trial function space, Petrov Weak Galerkin method is used. PWGFE method was defined in this section. Let  $\{T_h\}$  represent the family of triangulation shapes of which  $0 < h < 1$  represents.  $P_j(T^0)$  denotes the set of polynomials on  $T^0$  with a maximum degree of  $j$ , and  $P_\ell(\partial T)$  stands for the set of polynomials on  $\partial T$  with a maximum degree of  $\ell$ . A weak function  $v = \{v^0, v^b\}$  where  $v^0 \in P_j(T^0)$  and  $v^b \in P_\ell(\partial T)$  with  $j \geq 0$  and  $\ell \geq 0$  is called a discrete weak function  $v = \{v^0, v^b\}$ .

The discrete weak function space indicated by  $W(T, j, \ell)$  is

$$W(T, j, \ell) := \{v = \{v^0, v^b\}: v^0 \in P_j(T^0), v^b \in P_\ell(\partial T)\}. \quad (4.1)$$

Patching across all of the triangles  $T \in T_h$  would define the corresponding finite element space. Equivalently said, the weak finite element space is defined as

$$S_h(j, \ell) = \{v = \{v^0, v^b\}: \{v^0, v^b\}|_T \in W(T, j, \ell), \forall T \in T_h\}. \quad (4.2)$$

The  $S_h(j, \ell)$ 's subspace with vanishing boundary values on  $\partial\Omega$  is denoted as  $S_h^0(j, \ell)$ .

$$\text{i. e., } S_h^0(j, \ell) = \{v = \{v^0, v^b\} \in S_h(j, \ell), v^b|_{\partial T \cap \partial\Omega} = 0, \forall T \in T_h\}. \quad (4.3)$$

Now, we define the Petrov weak Galerkin spaces.

Let  $\psi_r$  be trail space and  $\phi_r$  be a test space which are define

$$\psi_r = \{v = \{v^0, v^b\} \in W(T, j, \ell), \forall T \in T_h\}$$

$$\phi_r = \{\rho: \rho = v^0 + \delta \beta \cdot \nabla_{d,r} v; v \in \psi_r\}$$

and

$$\psi_r^0 = \{v = \{v^0, v^b\} \in \psi_r, v^b|_{\partial T \cap \partial\Omega} = 0, \forall T \in T_h\}$$

$$\phi_r^0 = \{\rho : \rho = v^0 + \delta \beta \cdot \nabla_{d,r} v ; v \in \psi_r^0\} \quad (4.4)$$

In this context, "  $\delta$  " symbolize a constant stability parameter. It'll be designated as (Perella A. J. (1996).

$$\delta = \begin{cases} \xi h & \text{if } \epsilon < h \\ 0 & \text{if } \epsilon \geq h \end{cases} ; 0 < \xi < \frac{1}{4}, (\text{small constant}). \text{and } \dim \psi_r = \dim \phi_r$$

## 5. Weak Variational Formulation:

The Weak Variational Formulation for equation (1.1-1.3) is as follows: Find  $u \in \{u^0, u^b\} \in W(T, j, \ell)$  such that

$$(u_{0,t}, v^0) + (au(u) \nabla_d u \nabla_d v) - (u^0 b(u^0) \cdot \nabla_d v) - (u^0 \nabla_d b(u), v^0) + (cu^0, v^0) = (f, v^0) \\ (u^0(x, 0), v) = (u^0, v^0) \quad (5.1)$$

## 6. The Fully – Discrete Petrov Weak Galerkin (PWG) Variational Formulation:

In equation (1.3) we replace  $v$  with  $w$  then the fully discrete PWG Find  $u_h^N \in \psi_r$

$$(\bar{\partial} u_h^N, v^0) + (\bar{\partial} u_h^N, \delta \beta \cdot \nabla_{d,r} v) + (a(u) \nabla_{d,r} u, \nabla_{d,r} v) + (c u_h^N, v^0) + (c u_h^N, \delta \beta \cdot \nabla_{d,r} v) \\ - (u_h^N \nabla_{d,r} b(u), v^0) - (u_h^N \nabla_{d,r} b(u), \delta \beta \cdot \nabla_{d,r} v) - (u_h^N b(u_h), \nabla_{d,r} v) \\ (f^N, v^0) + (f^N, \delta \beta \cdot \nabla_{d,r} v) \quad , \forall v = \{v^0, v^b\} \in \psi_r^0 \quad (6.1)$$

The time space  $[0, T]$  in a fully-discrete scheme is thus uniformly discretized by  $n+1$  points  $t_n = N\tau, N = 0, 1, \dots, n$ , where the time step size is  $\tau = T/n$ . For any  $N \geq 1$ , a fully-discrete Petrov weak Galerkin finite element method is constructed as follows. The Crank-Nicolson Petrov weak Galerkin finite element solution. Find  $u_h^{N+1} = (u_h^{N+1,0}, u_h^{N+1,b}) \in \psi_r$ . Fulfilling

$$\left( \frac{u_h^{N+1,0} - u_h^{N,0}}{\tau}, v^0 \right) + \frac{u_h^{N+1,0} - u_h^{N,0}}{\tau}, \delta \beta \cdot \nabla_{d,r} v + \left( a(u_h^{N,0}) \nabla_{d,r} \frac{u_h^{N+1} + u_h^N}{2}, \nabla_{d,r} v \right) \\ + \left( c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, v^0 \right) + \left( c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} v \right) \\ - \left( \frac{u_h^{N+1,0} + u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v \right) - \left( \frac{u_h^{N+1,0} + u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0 \right)$$

$$-\left(\frac{u_h^{N+1,0}+u_h^{N,0}}{2}\nabla_{d,r}b(u_h^N),\delta\beta\cdot\nabla_{d,r}v\right)=\left(\frac{f^{N+1}+f^N}{2},v^0\right)+\left(\frac{f^{N+1}+f^N}{2},\delta\beta\cdot\nabla_{d,r}v\right),\forall v=\{v^0,v^b\}\in\psi_r^0 \quad (6.2)$$

**6.1. Stability:** In this section, we establish stability the Crank-Nicolson- Petrov weak Galerkin finite element's stability by means of the following lemma.

**Lemma 6.1:** for the solution  $u_h^{N+1}$  of equation (6.2) with a positive constant  $k_1$ , then

$$\|u_h^{N+1}\|^2 \leq k_1 h [(\tau + \tau^2)\|u_h^{N+1}\|^2 + (1 + \tau^2 + \tau)\|u_h^{N,0}\|^2 + \|f^{N+1} + f^N\|^2 + \tau^2\|\nabla_{d,r}u_h^N\|^2 + \|B\cdot\nabla_{d,r}u_h^{N+1}\|^2] \quad (6.3)$$

**Proof:** In equation (6.2) select  $v = u_h^{N+1}$

$$\begin{aligned} & (u_h^{N+1,0} - u_h^{N,0}, u_h^{N+1,0}) + (u_h^{N+1,0} - u_h^{N,0}, \delta B \nabla_{d,r} u_h^{N+1,0}) \\ & + [\tau(a(u_h^{N,0})\nabla_{d,r}\left(\frac{u_h^{N+1,0} + u_h^{N,0}}{2}\right), \nabla_{d,r}u_h^{N+1})] \\ & + \left(\tau c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, u_h^{N+1,0}\right) + \left(\tau c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, \delta B \nabla_{d,r} u_h^{N+1,0}\right) \\ & - \left(\tau \frac{u_h^{N+1,0} + u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} u_h^{N+1}\right) - \left(\tau \frac{u_h^{N+1,0} + u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), u_h^{N+1}\right) \\ & - \left(\tau \frac{u_h^{N+1,0} + u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta B \nabla_{d,r} u_h^{N+1}\right) = \left(\tau \frac{f^{N+1} + f^N}{2}, u_h^{N+1,0}\right) \\ & + \left(\tau \frac{f^{N+1} + f^N}{2}, \delta B \nabla_{d,r} u_h^{N+1,0}\right) \end{aligned}$$

Suppose the above equations is  $\sum_{i=1}^5 E^{(i)} = E^{(9)}$

Using the inequality  $((\gamma - \alpha, \gamma) \geq \frac{1}{2}(\|\gamma\|^2 - \|\alpha\|^2))$  [10], we have

$$|E^1| \geq \frac{1}{2}(\|u_h^{N+1,0}\|^2 - \|u_h^{N,0}\|^2).$$

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$|E^2| \leq (\|u_h^{N+1}\|^2 + \frac{1}{4}\delta\|B\nabla_{d,r}u_h^{N+1}\|^2) - (\|u_h^N\|^2 + \frac{1}{4}\delta\|B\nabla_{d,r}u_h^{N+1}\|^2)$$

By (Abdullah and Kashkool 2023)  $|E^3| \leq k h^{-1} \|u_h^{N+1}\|^2 + \tau^2 \frac{\|a(u_h^{N,0})\|^2}{16} \|\nabla_{d,r} u_h^{N+1}\|^2$ .

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$\begin{aligned} |E^4| &\leq \gamma_1(\tau + \tau^2) \|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2, \\ |E^5| &\leq \tau c \|u_h^{N+1,0}\|^2 + \frac{1}{4} \delta \|B \nabla_{d,r} u_h^{N+1}\|^2 + \tau c \|u_h^{N,0}\|^2 + \frac{1}{4} \delta \|B \nabla_{d,r} u_h^{N+1}\|^2 \\ |E^6| &\leq \gamma_2 h^{-1} \|u_h^{N+1}\|^2 + \tau^2 \|u_h^{N+1,0}\|^2 + \tau^2 \|u_h^{N,0}\|^2 \\ |E^7| &\leq \gamma_3(\tau + \tau^2) \|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2 \\ |E^8| &\leq \gamma_4(\tau + \tau^2) \delta \|B \nabla_{d,r} u_h^{N+1}\|^2 + \|u_h^{N+1,0}\|^2 + \|u_h^{N,0}\|^2 \\ |E^9| &\leq \frac{\tau^2}{16} \|u_h^{N+1,0}\|^2 + \|f^{N+1} + f^N\|^2 + \frac{\tau^2}{64} \cdot \delta \|B \nabla_{d,r} u_h^{N+1}\|^2 + \|f^{N+1} + f^N\|^2 \end{aligned}$$

Then the above equation becomes

$$\begin{aligned} (k + \gamma_2) h^{-1} \|u_h^{N+1}\|^2 &\leq \left( \frac{15\tau^2}{16} - \frac{1}{2} - \gamma_3(\tau + \tau^2) - \tau c \right) \|u_h^{N+1,0}\|^2 + 2 \|f^{N+1} + f^N\|^2 \\ &\quad - (\gamma_4(\tau + \tau^2) \delta + \frac{\delta}{8} + \frac{\delta \tau^2}{16}) \|B \nabla_{d,r} u_h^{N+1}\|^2 - \tau^2 \frac{\|a(u_h^{N,0})\|^2}{16} \|\nabla_{d,r} u_h^{N+1}\|^2 - \left( \frac{3}{2} + \tau^2 + \tau \right) \|u_h^{N,0}\|^2 \end{aligned}$$

Then

$$\begin{aligned} \|u_h^{N+1}\|^2 &\leq k_1 h [ (\tau + \tau^2 + 1) \|u_h^{N+1,0}\|^2 + (1 + \tau^2 + \tau) \|u_h^{N,0}\|^2 + \|f^{N+1} + f^N\|^2 \\ &\quad + \tau^2 \|\nabla_{d,r} u_h^N\|^2 + \|B \cdot \nabla_{d,r} u_h^{N+1}\|^2 \end{aligned}$$

Where  $k_1 = \max \left\{ -\frac{1}{2}, \frac{15}{16}, -\gamma_3, -c, 2, -\frac{\delta}{8}, -\frac{\delta}{16}, -\frac{\|a(u_h^{N,0})\|^2}{16}, -\frac{3}{2} \right\} / (k + \gamma_2)$  ■

## 7. Error Analysis of the Fully-Discrete Petrov Weak Galerkin Finite Element Scheme.

In this section, we obtain an error estimates for the Petrov weak Galerkin finite element method (6.2). we begin by determining an error equation for a weak Galerkin approximation  $u_h^{N+1}$  using the Crank-Nicolson method. In the Petrov weak Galerkin finite element space  $\psi_r$ ,  $h$  and the  $L^2$  projection of the exact solution  $u_h^{N+1}$  will demonstrate the error estimate for the  $L^2$  norm and  $H^1$  norm, and consequently, the optimal order error estimate for each.

### 7.1: The Error Equation of the Fully – Discrete Scheme.

An error equation is given by the following Lemma 6.1. for analyzing the difference between an  $L^2$  projection of the exact solution  $u_h^{N+1}$  of the original problem and a numerical solution  $u_h^{N+1}$  from the Petrov weak Galerkin finite element approximation. Let  $\theta^{N+1,0} = Q_h u^{N+1} - u_h^{N+1} = \{Q_h u^{N+1,0} - u_h^{N+1,0}, Q_h u^{N+1,b} - u_h^{N+1,b}\} = \{\theta^{N+1,0}, \theta^{N+1,b}\}$

**Lemma 7.1:** Let  $u_h^{N+1}$  be the solution of (6.2) then for each Let  $v \in \psi_r^0$  the error equation is

$$\begin{aligned} & \left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, v^0 \right) + \left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \delta \beta \nabla_{d,r} v \right) \\ & + (a(u_h^{N,0}) \nabla_{d,r} \left( \frac{\theta^{N+1,0} + \theta^N}{2}, \nabla_{d,r} v \right) + \left( c \frac{\theta^{N+1,0} + \theta^N}{2}, v^0 \right) \\ & + \left( c \frac{\theta^{N+1,0} + \theta^N}{2}, \delta \beta \nabla_{d,r} v \right) - \frac{(\theta^{N+1,0} + \theta^N b(u_h^{N,0}), \nabla_{d,r} v)}{2} \\ & - \left( \frac{\theta^{N+1,0} + \theta^N}{2} \nabla_{d,r} b(u_h^N), v^0 \right) - \left( \frac{\theta^{N+1,0} + \theta^N}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v \right) = \\ & \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, v^0 \right) + \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \cdot \nabla_{d,r} v \right), \quad (7.1) \end{aligned}$$

Proof : Using the operator projection  $Q_h$  from the bilinear equation (6.2)

$$\begin{aligned} & \sum_{(T,j,\ell) \in T_h} (a(u_h^{N,0}) \nabla_{d,r} \frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r} v)_{(T,j,\ell)} + \left( c \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0 \right)_{(T,j,\ell)} \\ & + \left( c \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, \delta \beta \nabla_{d,r} v \right)_{(T,j,\ell)} - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v \right)_{(T,j,\ell)} \\ & - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0 \right)_{(T,j,\ell)} - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v \right)_{(T,j,\ell)} \\ & = \left( a(u_h^{N,0}) \nabla_{d,r} \frac{u_h^{N+1} - u_h^N}{2}, v \right) - \left( \nabla_{d,r} \frac{u_h^{N+1,0} - u_h^{N,0}}{2} b(u_h^{N,0}), v \right) \\ & - \left( \frac{u_h^{N+1,0} - u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0 \right) - \left( \frac{u_h^{N+1,0} - u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v \right) \\ & + \left( c \frac{u_h^{N+1,0} - u_h^{N,0}}{2}, v^0 \right) + \left( c \frac{u_h^{N+1,0} - u_h^{N,0}}{2}, \delta \beta \nabla_{d,r} v \right) \end{aligned}$$

$$= -\left(\frac{\partial u_h^{N+1,0}}{\partial t}, v^0\right) - \left(\frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \nabla_{d,r} v\right) + \left(\frac{f^{N+1} + f^N}{2}, v^0\right) + \left(\frac{f^{N+1} + f^N}{2}, \delta \beta \nabla_{d,r} v\right)$$

Replacing  $\left(\frac{f^{N+1} + f^N}{2}, v^0\right) + \left(\frac{f^{N+1} + f^N}{2}, \delta \beta \nabla_{d,r} v\right)$  in the equation (6.2) obtain

$$\begin{aligned} & \sum_{(T,j,\ell) \in T_h} \left( a(u_h^{N,0}) \nabla_{d,r} \frac{Q_h u^{N+1} + Q_h u^N}{2}, \nabla_{d,r} v \right)_{(T,j,\ell)} + \left( c \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, v^0 \right)_{(T,j,\ell)} \\ & + \left( c \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2}, \delta \beta \nabla_{d,r} v \right)_{(T,j,\ell)} - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v \right)_{(T,j,\ell)} \\ & - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0 \right)_{(T,j,\ell)} - \left( \frac{Q_h u^{N+1,0} + Q_h u^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v \right)_{(T,j,\ell)} \\ & = -\left(\frac{\partial u_h^{N+1,0}}{\partial t}, v^0\right) - \left(\frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \nabla_{d,r} v\right) + \left(\frac{u_h^{N+1,0} - u_h^{N,0}}{\tau}, v^0\right) \\ & + \left(\frac{u_h^{N+1,0} - u_h^{N,0}}{\tau}, \delta \beta \nabla_{d,r} v\right) + \left(a(u_h^{N,0}) \nabla_{d,r} \frac{u_h^{N+1} - u_h^N}{2}, \nabla_{d,r} v\right) \\ & - \left(\frac{u_h^{N+1,0} + u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v\right) - \left(\frac{u_h^{N+1,0} + u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0\right) \\ & - \left(\frac{u_h^{N+1,0} + u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v\right) + \left(c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, v^0\right) \\ & + \left(c \frac{u_h^{N+1,0} + u_h^{N,0}}{2}, \delta \beta \nabla_{d,r} v\right) \end{aligned}$$

which leads to the error equation for the Crank-Nicolson-Petrov weak Galerkin scheme (6.2):

$$\begin{aligned} & \left(\frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, v^0\right) + \left(\frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \delta \beta \nabla_{d,r} v\right) \\ & + \left(a(u_h^{N,0}) \nabla_{d,r} \left(\frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \nabla_{d,r} v\right) + \left(c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, v^0\right) \right. \\ & + \left. \left(c \frac{\theta^{N+1,0} + \theta^{N,0}}{\tau}, \delta \beta \nabla_{d,r} v\right) - \left(\frac{\theta^{N+1,0} + \theta^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v\right) \right. \\ & - \left. \left(\frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0\right) - \left(\frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} v\right) = \right. \\ & \left. \left(\frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, v^0\right) + \left(\frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \cdot \nabla_{d,r} v\right) \blacksquare \right. \end{aligned}$$

**Lemma 7.2:** (Abdullah and Kashkool , 2023) For any  $u$  satisfying  $\frac{\partial^2 u}{\partial t^2} \in P_j(T^0)$  , we have

$$\left| \frac{u^{N+1} - u^N}{\tau} - \frac{\partial u^N}{\partial t} \right| \leq \frac{\tau}{12} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds,$$

## 7.2: The fully-discrete Petrov weak Galerkin finite element scheme's $L^2$ error estimates.

**Lemma 7.3:** Let  $\theta^{N+1}$  be the error estimate equation (7.1) and  $\sigma_1$  be a constant. Then we get  $\|\theta^{N+1}\|^2 \leq \sigma_1 h [\|\theta^{N+1,0}\|^2 + (\tau^2 + \tau + \xi)] \|\theta^{N,0}\|^2 + \left(\tau^2 + \frac{3}{4}\right) \|\beta \nabla_{d,r} \theta^N\|^2 +$

$$(\tau^2 + \tau) \|\nabla_{d,r} \theta^N\|^2 + \frac{\tau}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds, \quad (7.3)$$

**Proof:** when we use  $v = \theta^N$  in the equation (7.1) we get

$$\begin{aligned} & \left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \theta^{N,0} \right) + \left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \delta \beta \nabla_{d,r} \theta^N \right) \\ & + (a(u_h^{N,0}) \nabla_{d,r} \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \nabla_{d,r} \theta^N \right) + \left( c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \theta^{N,0} \right) \\ & + \left( c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \delta \beta \nabla_{d,r} \theta^N \right) - \left( \frac{(\theta^{N+1,0} + \theta^N)}{2} b(u_h^{N,0}), \nabla_{d,r} \theta^N \right) \\ & - \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \theta^N \right) - \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} \theta^N \right) = \\ & \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \theta^{N,0} \right) + \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \cdot \nabla_{d,r} \theta^N \right) \end{aligned}$$

The equation above can be rewritten as follows, we obtain

$$\begin{aligned} & [(\theta^{N+1,0} - \theta^{N,0}), \theta^{N,0}] + [(\theta^{N+1,0} - \theta^{N,0}), \delta \beta \nabla_{d,r} \theta^N] \\ & + (\tau a(u_h^{N,0}) \nabla_{d,r} \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \nabla_{d,r} \theta^N \right) - \left( \tau \frac{(\theta^{N+1,0} + \theta^N)}{2} b(u_h^{N,0}), \nabla_{d,r} \theta^N \right) \\ & - \left( \tau \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \theta^N \right) - \left( \tau \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} \theta^N \right) \\ & + \left( \tau c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \theta^{N,0} \right) + \left( \tau c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \delta \beta \nabla_{d,r} \theta^N \right) = \\ & \left( \tau \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t} \right), \theta^{N,0} \right) + \left( \tau \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t} \right), \delta \beta \cdot \nabla_{d,r} \theta^N \right) \end{aligned}$$

Suppose the above equation is  $\sum_{i=1}^8 F^{(i)} = F^{(9)}$

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$\begin{aligned}
 |F^1| &\leq \frac{1}{4} \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 \\
 |F^2| &\leq \delta \|\theta^{N+1,0}\|^2 + \frac{1}{2} \|\beta \nabla_{d,r} \theta^N\|^2 + \delta \lambda \|\theta^{N,0}\|^2 \\
 |F^3| &\leq \frac{\tau^2}{16} \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\nabla_{d,r} \theta^N\|^2 + \frac{\lambda \tau}{16} \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 \\
 |F^4| &\leq \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 |F^5| &\leq \delta \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\beta \nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 + \delta \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 |F^6| &\leq \delta \frac{\tau^2}{16} \|c\|^2 \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\tau}{2} \|c\| \lambda \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 |F^7| &\leq \delta \frac{\tau^2}{16} \|c\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\theta^{N+1,0}\|^2 + \frac{\delta}{2} \tau \|c\| \lambda \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{1}{4} \|\theta^{N,0}\|^2
 \end{aligned}$$

by young's inequality and Lemma (7.2) we obtain,

$$\begin{aligned}
 |F^8| &\leq \frac{\tau}{12} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds + \frac{\tau^2}{4} \|\theta^{N,0}\|^2 \\
 |F^9| &\leq \frac{\tau}{12} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds + \frac{\tau^2 \delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2
 \end{aligned}$$

Then the above equation is

$$\begin{aligned}
 &\frac{1}{4} \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 + \delta \|\theta^{N+1,0}\|^2 + \frac{1}{2} \|\beta \nabla_{d,r} \theta^N\|^2 + \delta \lambda \|\theta^{N,0}\|^2 + \\
 &\frac{\tau^2}{16} \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\nabla_{d,r} \theta^N\|^2 + \frac{\lambda \tau}{16} \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \\
 &\frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 &\delta \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\beta \nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 + \delta \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 &\frac{\tau^2}{16} \|c\|^2 \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\tau}{2} \|c\| \lambda \|\theta^{N,0}\|^2 + \\
 &\delta \frac{\tau^2}{16} \|c\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\theta^{N+1,0}\|^2 + \frac{\delta}{2} \tau \|c\| \lambda \|\theta^{N,0}\|^2 + \frac{1}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2
 \end{aligned}$$

$$\leq \frac{\tau}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds + \frac{\tau^2}{4} \|\theta^{N,0}\|^2 + \frac{\tau^2 \delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2$$

So,

$$\begin{aligned} \|\nabla_{d,r} \theta^{N+1}\|^2 &\leq \left[ \frac{\tau^2}{4} - \frac{9}{4} - (\delta + 1)\lambda - \frac{\tau^2}{16} \|c\|^2 - (1 + \delta) \frac{\tau}{2} \|c\| \lambda \right] \\ &- \left( \frac{16 + 3\delta}{4} \right) \|\theta^{N+1,0}\|^2 + \left[ \delta \frac{\tau^2}{16} - \frac{3}{4} \delta - \delta \tau^2 \|b(u_h^{N,0})\|^2 - \frac{\tau^2 \delta}{16} \|b(u_h^{N,0})\|^2 \right. \\ &- \frac{\tau^2}{16} \|c\|^2 \left. \right] \|\beta \cdot \nabla_{d,r} \theta^N\|^2 - [\tau^2 \|a(u_h^{N,0})\|^2 + \frac{\lambda}{2} \|a(u_h^{N,0})\|^2 + \tau^2 \|b(u_h^{N,0})\|^2 + \\ &\frac{\tau^2 \delta}{16} \|b(u_h^{N,0})\|^2] \|\nabla_{d,r} \theta^N\|^2 \leq \frac{\tau}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds, \end{aligned}$$

Hence

$$\begin{aligned} \|\theta^{N+1}\|^2 &\leq \sigma_1 h [\|\theta^{N+1,0}\|^2 + (\tau^2 + \tau + \xi) \|\theta^{N,0}\|^2 + \left( \tau^2 + \frac{3}{4} \right) \|\beta \nabla_{d,r} \theta^N\|^2 + \\ &(\tau^2 + \tau) \|\nabla_{d,r} \theta^N\|^2 + \frac{\tau}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds, \end{aligned} \quad \blacksquare$$

Where,  $\sigma_1 = \max\{\frac{-16+3\delta}{4}, \frac{1}{4}, \frac{-9}{4}, -\lambda, -\frac{\|c\|}{16}, \frac{\delta}{4}, -\delta, -\delta \|b(u_h^{N,0})\|^2, -\frac{\delta}{16} \|b(u_h^{N,0})\|^2,$

$-\frac{\delta \|c\|^2}{16}, -\|a(u_h^{N,0})\|^2, -\frac{\lambda}{2} \|a(u_h^{N,0})\|^2, -\|b(u_h^{N,0})\|^2, \frac{1}{16} \|b(u_h^{N,0})\|^2\} / \sigma$  and  $\xi = (1 + \delta)$

### 7.3: The Fully-Discrete Petrov Weak Galerkin Finite Element's $H^1$ Error Estimates.

**Theorem 7.4:** Let  $\theta^{N+1}$  be the error estimate equation (7.1) and  $\sigma_1$  be a constant. Then we get,

$$\begin{aligned} \|\theta^{N+1}\|^2 + \|\nabla_{d,r} \theta^{N+1}\|^2 &\leq \sigma_1 h [\|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \|\beta \nabla_{d,r} \theta^{N+1,0}\|^2 + \|\nabla_{d,r} \theta^N\|^2 \\ &+ \|\beta \nabla_{d,r} \theta^N\|^2 + \frac{\tau}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds, \end{aligned} \quad (7.4)$$

**Proof:** when we use  $v = \frac{\theta^{N+1} - \theta^N}{\tau}$  in the equation (7.1) we get

$$\begin{aligned} &\left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau} \right) + \left( \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau}, \delta \beta \nabla_{d,r} \frac{\theta^{N+1} - \theta^N}{\tau} \right) \\ &+ (a(u_h^{N,0}) \nabla_{d,r} \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \nabla_{d,r} \frac{\theta^{N+1} - \theta^N}{\tau} \right) + \left( c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau} \right) + \\ &\left( c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \delta \beta \nabla_{d,r} \left( \frac{\theta^{N+1} - \theta^N}{\tau} \right) \right) - \left( \frac{(\theta^{N+1,0} + \theta^N)}{2} b(u_h^{N,0}), \nabla_{d,r} \left( \frac{\theta^{N+1} - \theta^N}{\tau} \right) \right) \end{aligned}$$

$$\begin{aligned}
 & - \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \frac{\theta^{N+1} - \theta^N}{\tau} \right) - \left( \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} \frac{\theta^{N+1} - \theta^N}{\tau} \right) = \\
 & \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \frac{\theta^{N+1,0} - \theta^{N,0}}{\tau} \right) + \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t}, \delta \beta \cdot \nabla_{d,r} \frac{\theta^{N+1} - \theta^N}{\tau} \right) \\
 & \text{rewritten the above equation} \\
 & ((\theta^{N+1,0} - \theta^{N,0}), (\theta^{N+1,0} - \theta^{N,0})) + (\theta^{N+1,0} - \theta^{N,0}, \delta \beta \cdot \nabla_{d,r} (\theta^{N+1} - \theta^N)) \\
 & \left( \tau a(u_h^{N,0}) \nabla_{d,r} \frac{\theta^{N+1} + \theta^N}{2}, \nabla_{d,r} (\theta^{N+1} - \theta^N) \right) + \left( \tau c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \theta^{N+1,0} - \theta^{N,0} \right) + \\
 & \left( \tau c \frac{\theta^{N+1,0} + \theta^{N,0}}{2}, \delta \beta \nabla_{d,r} (\theta^{N+1} - \theta^N) \right) - \left( \tau \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^{N,0}), (\theta^{N+1} - \theta^N) \right) \\
 & - \left( \tau \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^{N,0}), \theta^{N+1,0} - \theta^{N,0} \right) \\
 & - \left( \tau \frac{\theta^{N+1,0} + \theta^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \nabla_{d,r} \theta^{N+1} - \theta^N \right) = \\
 & \left( \tau \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t} \right), \theta^{N+1,0} - \theta^{N,0} \right) + \\
 & \left( \tau \left( \frac{Q_h u^{N+1,0} - Q_h u^{N,0}}{\tau} - \frac{\partial u_h^{N+1,0}}{\partial t} \right), \delta \beta \cdot \nabla_{d,r}, \theta^{N+1,0} - \theta^{N,0} \right)
 \end{aligned}$$

Suppose the above equation is  $\sum_{i=1}^8 F^{(i)} = F^{(9)} + F^{(10)}$  (7.5)

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$\begin{aligned}
 |F^1| & \leq \lambda \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 \\
 |F^2| & \leq \delta \|\theta^{N+1,0}\|^2 + \frac{1}{4} \delta \|\beta \nabla_{d,r} \theta^{N+1}\|^2 + \delta \lambda \|\theta^{N,0}\|^2 + \frac{1}{4} \delta \|\beta \nabla_{d,r} \theta^N\|^2 \\
 |F^3| & \leq \tau \lambda h^{-1} \|\theta^{N+1}\|^2 + \frac{1}{2} \tau \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 \\
 |F^4| & \leq \frac{\tau}{2} \|c\| \|\theta^{N+1,0}\|^2 + \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 \\
 |F^5| & \leq \delta \frac{\tau^2}{16} \|c\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\theta^{N+1,0}\|^2 + \frac{\delta}{2} \tau \|c\| \sigma \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{1}{4} \|\theta^{N,0}\|^2 \\
 |F^6| & \leq \frac{\tau}{16} \|b(u_h^{N,0})\|^2 \|\theta^{N+1,0}\|^2 + \tau \|\nabla_{d,r} \theta^N\|^2 + \frac{\tau}{16} \|b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2 + \tau \|\nabla_{d,r} \theta^{N,0}\|^2 \\
 |F^7| & \leq \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N+1,0}\|^2 + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2
 \end{aligned}$$

$$|F^8| \leq \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N+1,0}\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\ + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2$$

By using Young's – inequality and Lemma (7.2) we get;

$$|F^9| \leq \lambda \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 + \frac{\tau^3}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds \\ |F^{10}| \leq \lambda_1 \|\beta \cdot \nabla_{d,r} \theta^{N+1,0}\|^2 + \lambda_1 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\tau^3}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds$$

Then, from the above equation, we get;

$$\lambda \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 + \delta \|\theta^{N+1,0}\|^2 + \frac{1}{4} \delta \|\beta \nabla_{d,r} \theta^{N+1}\|^2 + \delta \lambda \|\theta^{N,0}\|^2 \\ + \frac{1\delta}{4} \|\beta \nabla_{d,r} \theta^N\|^2 + \tau \lambda h^{-1} \|\nabla_{d,r} \theta^{N+1}\|^2 + \frac{1}{2} \tau \|a(u_h^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \\ \frac{\tau}{2} \|c\| \|\theta^{N+1,0}\|^2 + \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 + \delta \frac{\tau^2}{16} \|c\|^2 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\theta^{N+1,0}\|^2 \\ + \frac{\tau\delta}{2} \|c\| \sigma \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\tau}{16} \|b(u_h^{N,0})\|^2 \|\theta^{N+1,0}\|^2 + \tau \|\nabla_{d,r} \theta^N\|^2 \\ + \frac{\tau}{16} \|b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2 + \tau \|\nabla_{d,r} \theta^{N,0}\|^2 + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N+1,0}\|^2 \\ + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2 + \frac{\tau}{2} \|b(u_h^N)\|^2 \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\ + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 \|\theta^{N,0}\|^2 \leq \lambda \|\theta^{N+1,0}\|^2 + \lambda \|\theta^{N,0}\|^2 + \frac{\tau^3}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds \\ + \lambda_1 \|\beta \cdot \nabla_{d,r} \theta^{N+1,0}\|^2 + \lambda_1 \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\tau^3}{6} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds$$

So,  $\lambda h^{-1} \|\nabla_{d,r} \theta^{N+1}\|^2 + \tau \|\nabla_{d,r} \theta^{N,0}\|^2 \leq \frac{\tau^3}{2} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds$

$$-(\lambda + \delta + \frac{\tau}{2} \|c\| + \delta \frac{\tau^2}{8} \|c\|^2 + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 + (\frac{\tau}{2} \|b(u_h^N)\|^2 + -\lambda) \|\theta^{N+1,0}\|^2$$

$$\begin{aligned}
 & -(\lambda + \frac{\tau}{2} \|c\| + \delta \frac{\tau}{2} \|c\|^2 \sigma + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 + \frac{\tau}{2} \|b(u_h^N)\|^2 + \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2 - \lambda) \\
 & \|\theta^{N,0}\|^2 - \left(\frac{5}{4} \delta - \lambda_1\right) \|\beta \cdot \nabla_{d,r} \theta^{N+1,0}\|^2 - \left(\frac{1}{2} \tau \|a(u_h^{N,0})\|^2 + \tau\right) \|\nabla_{d,r} \theta^N\|^2 \\
 & + \lambda_1 \|\beta \cdot \nabla_{d,r} \theta^N\|^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Then, } \|\nabla_{d,r} \theta^{N+1}\|^2 + \|\nabla_{d,r} \theta^{N,0}\|^2 & \leq \sigma_1 h [\|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \|\beta \cdot \nabla_{d,r} \theta^{N+1,0}\|^2 \\
 & + \|\nabla_{d,r} \theta^N\|^2 + \frac{\tau^3}{2} \int_{\tau_N}^{\tau_{N+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|^2 ds
 \end{aligned}$$

$$\text{Where, } \sigma_1 = \max\{-\delta, -\frac{\tau}{2} \|c\|, -(\delta \frac{\tau^2}{8} \|c\|^2 + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 + \frac{\tau}{2} \|b(u_h^N)\|^2),$$

$$\begin{aligned}
 & -\frac{\tau}{2} \|b(u_h^N)\|^2 + \lambda, -(\frac{\tau}{2} \|c\| + \delta \frac{\tau}{2} \|c\|^2 \sigma + \frac{\tau^2}{16} \|b(u_h^{N,0})\|^2 + \frac{\tau}{2} \|b(u_h^N)\|^2 + \\
 & \frac{\tau}{2} \|\nabla_{d,r} b(u_h^{N,0})\|^2), -\frac{5}{4} \delta, \lambda_1, -\left(\frac{1}{2} \tau \|a(u_h^{N,0})\|^2 + \tau\right), -\lambda_1
 \end{aligned}$$

#### 7.4: Estimates of the $L^2$ – Normal optimal order Error

In this subsection, we determined the optimal order error estimate in the  $L^2$ -norm for discrete time Petrov weak Galerkin finite element methods.

Let  $u^{N+1} \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\pi_h u^{N+1}$  denote the elliptic projection of  $u_h^{N+1}$  on to finite element space  $\psi_r^0$ .

**Theorem 7.6:** Assume that there is a constant  $\eta$  and that the precise solution of equation (7.2) is so regular that  $u^{N+1} \in H^{k+1}(\Omega)$ . Then

$$\begin{aligned}
 \|\theta^{N+1,0}\|^2 & \leq \eta h [(1 + \tau^2) \|\theta^{N+1,0}\|^2 + (1 + \tau) \|\theta^{N,0}\|^2 + (1 + \tau^2) \|\nabla_{d,r} \theta^N\|^2 + \\
 & \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \tau^2 \|\pi_h \nabla_{d,r} u^N\|^2 + \tau^2 \|\pi_h u^{N+1,0}\|^2 + (\tau^2 + \tau) \|\pi_h u^{N,0}\|^2] \quad (7.6)
 \end{aligned}$$

**Proof:** in equation (6.2), choose  $u^N = \pi_h u^N$  and  $u_h^N = Q_h u^N$  respectively

$$\begin{aligned}
 & \left( \frac{\pi_h u_h^{N+1,0} - \pi_h u_h^{N,0}}{\tau}, v^0 \right) + \left( \frac{\pi_h u_h^{N+1,0} - \pi_h u_h^{N,0}}{\tau}, \delta \beta \cdot \nabla_{d,r} v \right) + \\
 & \left( a(u) \pi_h \nabla_{d,r} \frac{u_h^{N+1} + u_h^N}{2}, \nabla_{d,r} v \right) + \left( c \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2}, v^0 \right) \\
 & + \left( c \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} v \right) - \left( \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2} b(u_h^{N,0}), \nabla_{d,r} v \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \left( \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), v^0 \right) - \left( \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2} \nabla_{d,r} b(u_h^N), \delta \beta \cdot \nabla_{d,r} v \right) \\
 & = \left( \frac{f^{N+1} + f^N}{2}, v^0 \right) + \left( \frac{f^{N+1} + f^N}{2}, \delta \beta \cdot \nabla_{d,r} v \right), \quad \forall v = \{v^0, v^b\} \in \psi_r^0 \quad (7.7)
 \end{aligned}$$

and,

$$\begin{aligned}
 & \left( \frac{Q_h u_h^{N+1,0} - Q_h u_h^{N,0}}{\tau}, v^0 \right) + \left( \frac{Q_h u_h^{N+1,0} - Q_h u_h^{N,0}}{\tau}, \delta \beta \cdot \nabla_{d,r} v \right) + \\
 & \left( a(u) \nabla_{d,r} \frac{Q_h u_h^{N+1} + Q_h u_h^N}{2}, \nabla_{d,r} v \right) + \left( c \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2}, v^0 \right) \\
 & + \left( c \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} v \right) - \left( \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2} b(u^N), \nabla_{d,r} v \right) \\
 & - \left( \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2} \nabla_{d,r} b(u), v^0 \right) - \left( \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2} \nabla_{d,r} b(u), \delta \beta \cdot \nabla_{d,r} v \right) \\
 & = \left( \frac{f^{N+1} + f^N}{2}, v^0 \right) + \left( \frac{f^{N+1} + f^N}{2}, \delta \beta \cdot \nabla_{d,r} v \right), \quad \forall v = \{v^0, v^b\} \in \psi_r^0 \quad (7.8)
 \end{aligned}$$

we get by subtracting equation (7.8) from (7.7)

$$\begin{aligned}
 & \left( \frac{\pi_h u_h^{N+1,0} - \pi_h u_h^{N,0}}{\tau}, v^0 \right) - \left( \frac{Q_h u_h^{N+1,0} - Q_h u_h^{N,0}}{\tau}, v^0 \right) + \\
 & \left( \frac{\pi_h u_h^{N+1,0} - \pi_h u_h^{N,0}}{\tau}, \delta \beta \cdot \nabla_{d,r} v \right) - \left( \frac{Q_h u_h^{N+1,0} - Q_h u_h^{N,0}}{\tau}, \delta \beta \cdot \nabla_{d,r} v \right) \\
 & + \left( a(\pi_h u^{N,0}) \pi_h \nabla_{d,r} \frac{u_h^{N+1} + u_h^N}{2}, \nabla_{d,r} v \right) - \left( a(Q_h u^{N,0}) \nabla_{d,r} \frac{Q_h u_h^{N+1} + Q_h u_h^N}{2}, \nabla_{d,r} v \right) + \\
 & \left( c \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2}, v^0 \right) - \left( c \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2}, v^0 \right) + \\
 & \left( c \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} v \right) - \left( c \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} v \right) \\
 & - \left( \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2} b(\pi_h u^{N,0}), \nabla_{d,r} v \right) + \left( \frac{Q_h u_h^{N+1,0} - Q_h u_h^{N,0}}{\tau} b(Q_h u^{N,0}), \nabla_{d,r} v \right) - \\
 & \left( \frac{\pi_h u_h^{N+1,0} + \pi_h u_h^{N,0}}{2} \nabla_{d,r} b(u^N), v^0 \right) - \left( \frac{\pi_h u_h^{N+1,0} - \pi_h u_h^{N,0}}{\tau} \nabla_{d,r} b(u^N), \delta \beta \cdot \nabla_{d,r} v \right) +
 \end{aligned}$$

$$\left( \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2} \pi_h \nabla_{d,r} b(u^N), v^0 \right) + \left( \frac{Q_h u_h^{N+1,0} + Q_h u_h^{N,0}}{2} \nabla_{d,r} b(u^N), \delta \beta \cdot \nabla_{d,r} v \right) = 0 \quad (7.9)$$

rewrite the equation (7.9) and use  $v = \theta^N = \pi_h u^N - Q_h u^N$ , we get;

$$\begin{aligned} & (\theta^{N+1,0}, \theta^{N,0}) - (\theta^{N,0}, \theta^{N,0}) + (\theta^{N+1,0}, \delta \beta \cdot \nabla_{d,r} \theta^N) - (\theta^{N,0}, \delta \beta \cdot \nabla_{d,r} \theta^N) + \\ & \left( \tau a(\pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^{N+1}}{2}, \nabla_{d,r} \theta^N \right) + \left( \tau c \frac{\theta^{N,0}}{2}, \theta^{N,0} \right) + \left( \tau c \frac{\theta^{N,0}}{2}, \delta \beta \cdot \nabla_{d,r} \theta^N \right) + \\ & \left( \tau c \frac{\theta^{N+1,0}}{2}, \theta^{N,0} \right) + \left( \tau c \frac{\theta^{N+1,0}}{2}, \delta \beta \cdot \nabla_{d,r} \theta^N \right) + \left( \tau a(\pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^N}{2}, \nabla_{d,r} \theta^N \right) \\ & + \left( \tau \left( \frac{a(\pi_h u^{N,0}) - a(Q_h u^{N,0})}{2} \right) \pi_h \nabla_{d,r} u^{N+1}, \nabla_{d,r} \theta^N \right) + \\ & \left( \tau \left( \frac{a(\pi_h u^{N,0}) - a(Q_h u^{N,0})}{2} \right) \pi_h \nabla_{d,r} u^N, \nabla_{d,r} \theta^N \right) - \\ & \left( \tau \frac{\theta^{N+1,0}}{2} b(\pi_h u^{N,0}), \nabla_{d,r} \theta^N \right) - \left( \tau \frac{\theta^N}{2} b(\pi_h u^{N,0}), \nabla_{d,r} \theta^N \right) - \\ & \left( \tau \frac{(b(\pi_h u^{N,0}) - b(Q_h u^{N,0}))}{2} \pi_h u^{N+1,0}, \nabla_{d,r} \theta^N \right) \\ & \left( \tau \frac{(b(\pi_h u^{N,0}) - b(Q_h u^{N,0}))}{2} \pi_h u^{N,0}, \nabla_{d,r} \theta^N \right) - \left( \tau \frac{\theta^{N+1,0}}{2} \pi_h \nabla_{d,r} b(u^N), \theta^{N,0} \right) \\ & - \left( \tau \frac{\theta^{N+1,0}}{2} \pi_h \nabla_{d,r} b(u^N), \delta \beta \cdot \nabla_{d,r} \theta^N \right) - \left( \tau \frac{\theta^{N,0}}{2} \pi_h \nabla_{d,r} b(u^N), \theta^{N,0} \right) - \\ & \left( \tau \frac{\theta^{N,0}}{2} \pi_h \nabla_{d,r} b(u^N), \delta \beta \cdot \nabla_{d,r} \theta^N \right) - \left( \tau \nabla_{d,r} \frac{(b(\pi_h u^N) - b(Q_h u^N))}{2} \pi_h u^{N+1,0}, \theta^{N,0} \right) \\ & - \left( \tau \nabla_{d,r} \frac{(b(\pi_h u^N) - b(Q_h u^N))}{2} \pi_h u^{N+1,0}, \delta \beta \cdot \nabla_{d,r} \theta^N \right) - \\ & \left( \tau \nabla_{d,r} \frac{(b(\pi_h u^N) - b(Q_h u^N))}{2} \pi_h u^{N,0}, \theta^{N,0} \right) \\ & - \left( \tau \nabla_{d,r} \frac{(b(\pi_h u^{N,0}) - b(Q_h u^{N,0}))}{2} \pi_h u^{N,0}, \delta \beta \cdot \nabla_{d,r} \theta^N \right) \quad (7.10) \end{aligned}$$

Suppose the above equation is  $\sum_{i=1}^{20} F^{(i)} = 0$ , (7.11)

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$F^1 \leq \|\theta^{N+1,0}\|^2 + \frac{1}{4} \|\theta^{N,0}\|^2, \quad F^2 \leq \xi \|\theta^{N,0}\|^2$$

$$\begin{aligned}
 F^3 &\leq \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2, F^4 \leq \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 F^5 &\leq \alpha_1 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \lambda h^{-1} \|\theta^{N+1,0}\|^2, F^6 \leq \frac{\tau^2}{16} \|c\|^2 \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 \\
 F^7 &\leq \frac{\tau^2}{16} \|c\|^2 \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2, F^8 \leq \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 \\
 F^9 &\leq \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2, F^{10} \leq \alpha_2 \tau^2 \|\pi_h \nabla_{d,r} u^N\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2 \\
 F^{11} &\leq \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2, \\
 F^{12} &\leq \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 \\
 F^{13} &\leq \alpha_3 \tau^2 \|\pi_h u^{N+1,0}\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2 \\
 F^{14} &\leq \alpha_4 \tau^2 \|\pi_h u^{N,0}\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2, F^{15} \leq \xi \tau^2 \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 \\
 F^{16} &\leq \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^N)\|^2}{16} \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 F^{17} &\leq \alpha_5 \tau^2 \|\pi_h u^{N+1,0}\|^2 + 2 \|\theta^{N,0}\|^2 \\
 F^{18} &\leq \alpha_6 \tau^2 \|\pi_h u^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 F^{19} &\leq \alpha_7 \tau^2 \|\pi_h u^{N,0}\|^2 + 2 \|\theta^{N,0}\|^2 \\
 F^{20} &\leq \tau^2 \left( \frac{\|\nabla_{d,r} b(\pi_h u^N)\|^2}{16} \right) \|\pi_h u^{N,0}\|^2 + \frac{\delta}{2} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \tau^2 \left( \frac{\|\nabla_{d,r} b(Q_h u^N)\|^2}{16} \right) \|\pi_h u^{N,0}\|^2
 \end{aligned}$$

Then the above equation becomes, we obtain;

$$\begin{aligned}
 &\|\theta^{N+1,0}\|^2 + \frac{1}{4} \|\theta^{N,0}\|^2 + \xi \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \\
 &\|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \alpha_1 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \lambda h^{-1} \|\theta^{N+1,0}\|^2 + \frac{\tau^2}{16} \|c\|^2 \|\theta^{N+1,0}\|^2 \\
 &+ \|\theta^{N,0}\|^2 + \frac{\tau^2}{16} \|c\|^2 \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 +
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\tau}{2} \|c\| \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \alpha_2 \tau^2 \|\pi_h \nabla_{d,r} u^N\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2 + \\
 & \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 + \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N,0}\|^2 + \\
 & \alpha_3 \tau^2 \|\pi_h u^{N+1,0}\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2 + \alpha_4 \tau^2 \|\pi_h u^{N,0}\|^2 + 2 \|\nabla_{d,r} \theta^N\|^2 + \\
 & \xi \tau^2 \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^N)\|^2}{16} \|\theta^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 & + \alpha_5 \tau^2 \|\pi_h u^{N+1,0}\|^2 + 2 \|\theta^{N,0}\|^2 + \alpha_6 \tau^2 \|\pi_h u^{N+1,0}\|^2 + \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 & + \alpha_7 \tau^2 \|\pi_h u^{N,0}\|^2 + 2 \|\theta^{N,0}\|^2 + \tau^2 \left( \frac{\|\nabla_{d,r} b(\pi_h u^N)\|^2}{16} \right) \|\pi_h u^{N,0}\|^2 + \frac{\delta}{2} \|\beta \cdot \nabla_{d,r} \theta^N\|^2 \\
 & + \tau^2 \left( \frac{\|\nabla_{d,r} b(Q_h u^N)\|^2}{16} \right) \|\pi_h u^{N,0}\|^2 \leq 0
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } \lambda h^{-1} \|\theta^{N+1}\|^2 & \leq - \left( 3 + \frac{\tau^2}{8} \|c\|^2 + \xi \tau^2 + \tau^2 \|\pi_h \nabla_{d,r} b(u^N)\|^2 \right) \|\theta^{N+1,0}\|^2 \\
 & - \left( \frac{41}{4} + \xi + \tau \|c\| \right) \|\theta^{N,0}\|^2 - (\alpha_1 \tau^2 + 6 + \tau^2 \|b(\pi_h u^{N,0})\|^2) \|\nabla_{d,r} \theta^N\|^2 - \\
 & \left( \frac{7\delta}{4} \right) \|\beta \cdot \nabla_{d,r} \theta^N\|^2 - (\alpha_2 \tau^2 \|\pi_h \nabla_{d,r} u^N\|^2 - ((\alpha_3 + \alpha_6) \tau^2) \|\pi_h \nabla_{d,r} u^{N+1,0}\|^2 - \\
 & (\alpha_4 \tau^2 + \alpha_7 \tau + \tau^2 \frac{\|\nabla_{d,r} b(\pi_h u^{N,0})\|^2}{16} + \tau^2 \frac{\|\nabla_{d,r} b(Q_h u^{N,0})\|^2}{16}) \|\pi_h u^{N,0}\|^2
 \end{aligned}$$

Then

$$\begin{aligned}
 \|\theta^{N+1}\|^2 & \leq \eta h [(1 + \tau^2) \|\theta^{N+1,0}\|^2 + (1 + \tau) \|\theta^{N,0}\|^2 + (1 + \tau^2) \|\nabla_{d,r} \theta^N\|^2 + \\
 & \|\beta \cdot \nabla_{d,r} \theta^N\|^2 + \tau^2 \|\pi_h \nabla_{d,r} u^N\|^2 + \tau^2 \|\pi_h u^{N+1,0}\|^2 + (\tau^2 + \tau) \|\pi_h u^{N,0}\|^2] \blacksquare
 \end{aligned}$$

Where,  $\eta = \max\{-3, -\frac{\|c\|^2}{8}, -\xi, -\|b(\pi_h \nabla_{d,r} u^{N,0})\|^2, -\frac{41}{4}, -\frac{\|a(\pi_h u^{N,0})\|^2}{2}, -\xi, -\|c\|, -\alpha_1,$

$$-6, -\|b(\pi_h u^{N,0})\|^2, -\frac{7\delta}{4}, -\alpha_2, -(\alpha_3 + \alpha_6), -(\alpha_3 + \alpha_6 + \|\nabla_{d,r} b(\pi_h u^{N,0})\|^2 + \|\nabla_{d,r} b(Q_h u^{N,0})\|^2)\} / \lambda$$

## 7.5 Estimates of the Optimal Order Error of the $H^1$ Norm.

In this subsection, we determined the optimal order error estimate in the  $H^1$  norm for discrete time Petrov weak Galerkin finite element methods.

Assume that  $u^{N+1} \in H_0^1(\Omega) \cap H^2(\Omega)$  and  $\pi_h u^{N+1}$  represent the elliptic projection of  $u_h^{N+1}$  on to  $\psi_r^0$  finite element space.

**Theorem 7.7:** Assume that a constant  $\eta_1$  exists and that the exact solution of equation (6.2) is so regular that  $u^{N+1} \in H^{K+1}(\Omega)$  then

$$\begin{aligned} \|\theta^{N+1,0}\|^2 + \|\nabla_{d,r}\theta^N\|^2 &\leq \eta_1 h[\tau^2(\|\pi_h \nabla_{d,r} u^{N+1}\|^2 + \|\nabla_{d,r} u_h^N\|^2 + [1 + \tau + \tau^2]\|\theta^{N+1,0}\|^2 \\ &+ \tau^2\|\theta^{N,0}\|^2 + \|\beta \cdot \nabla_{d,r}\theta^{N+1}\|^2 + \tau^2\|\nabla_{d,r}\theta^{N,0}\|^2 + (1 + \tau^2)\|\theta^{N,0}\|^2], \quad (7.12) \end{aligned}$$

**Proof:** Rewrite equation (7.10) and using  $v = \theta^{N+1,0} = \pi_h u^{N+1} - Q_h u^{N+1}$ , we obtain;

$$\begin{aligned} &(\theta^{N+1,0}, \theta^{N+1,0}) - (\theta^{N,0}, \theta^{N+1,0}) + (\theta^{N+1,0}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}) - (\theta^{N,0}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}) + \\ &\left(\tau a(\pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^{N+1}}{2}, \nabla_{d,r}\theta^{N+1}\right) + \left(\tau c \frac{\theta^{N,0}}{2}, \theta^{N+1,0}\right) + \left(\tau c \frac{\theta^{N,0}}{2}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) + \\ &\left(\tau c \frac{\theta^{N+1,0}}{2}, \theta^{N+1,0}\right) + \left(\tau c \frac{\theta^{N+1,0}}{2}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) + \left(\tau a(\pi_h u^{N,0}) \nabla_{d,r} \frac{\theta^N}{2}, \nabla_{d,r}\theta^{N+1}\right) \\ &+ \left(\tau \frac{a(\pi_h u^{N,0}) - a(Q_h u^{N,0})}{2} \pi_h \nabla_{d,r} u^{N+1}, \nabla_{d,r}\theta^{N+1}\right) + \\ &\left(\tau \frac{a(\pi_h u^{N,0}) - a(Q_h u^{N,0})}{2} \pi_h \nabla_{d,r} u^{N,0}, \nabla_{d,r}\theta^{N+1}\right) - \left(\tau \frac{\theta^{N+1,0}}{2} b(\pi_h u^{N,0}), \nabla_{d,r}\theta^{N+1}\right) - \\ &\left(\tau \frac{\theta^{N,0}}{2} b(\pi_h u^{N,0}), \nabla_{d,r}\theta^{N+1}\right) - \left(\tau \frac{b(\pi_h u^{N,0}) - b(Q_h u^{N,0})}{2} \pi_h \nabla_{d,r} u^{N+1,0}, \nabla_{d,r}\theta^{N+1}\right) - \\ &\left(\tau \frac{b(\pi_h u^{N,0}) - b(Q_h u^{N,0})}{2} \pi_h \nabla_{d,r} u^{N,0}, \nabla_{d,r}\theta^{N+1}\right) - \left(\tau \frac{\theta^{N+1,0}}{2} \pi_h \nabla_{d,r} b(u^N), \theta^{N+1,0}\right) - \\ &\left(\tau \frac{\theta^{N+1,0}}{2} \pi_h \nabla_{d,r} b(u^N), \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) - \left(\tau \frac{\theta^{N,0}}{2} \pi_h \nabla_{d,r} b(u^N), \theta^{N+1,0}\right) \\ &- \left(\tau \frac{\theta^{N,0}}{2} \pi_h \nabla_{d,r} b(u^N), \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) - \left(\tau \nabla_{d,r} \frac{b(\pi_h u^N) - b(Q_h u^N)}{2} \pi_h u^{N+1,0}, \theta^{N+1,0}\right) \\ &\left(\tau \nabla_{d,r} \frac{b(\pi_h u^N) - b(Q_h u^N)}{2} \pi_h u^{N+1,0}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) - \\ &\left(\tau \nabla_{d,r} \frac{b(\pi_h u^N) - b(Q_h u^N)}{2}, \theta^{N+1,0}\right) - \left(\tau \nabla_{d,r} \frac{(b(\pi_h u^{N,0}) - b(Q_h u^{N,0}))}{2}, \delta\beta \cdot \nabla_{d,r}\theta^{N+1}\right) \end{aligned}$$

Suppose the above equation is  $\sum_{i=1}^{23} F^{(i)} = 0$ , (7.13)

By using by Cauchy – Schwartz inequality and Young's - inequality, we get;

$$F^1 \leq \rho \|\theta^{N+1,0}\|^2, F^2 \leq \frac{1}{4} \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2$$

$$\begin{aligned}
 F^3 &\leq \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2, F^4 \leq \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^5 &\leq \frac{\tau}{2} \|a(\pi_h u^{N,0})\| \|\nabla_{d,r} \theta^{N+1}\|^2 + \frac{\tau^2}{16} \|a(\pi_h u^{N,0})\|^2 \|\nabla_{d,r} \theta^N\|^2 + \|\nabla_{d,r} \theta^N\|^2 \\
 F^6 &\leq \frac{\tau^2}{16} \|c\|^2 \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2, F^7 \leq \frac{\tau^2}{16} \|c\|^2 \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^8 &\leq \frac{\tau}{2} \|c\| \|\theta^{N+1,0}\|^2, F^9 \leq \frac{\tau}{2} \|c\| \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{10} &\leq \Theta_1 \tau^2 \|\pi_h \nabla_{d,r} u^{N+1}\|^2 + 2\lambda h^{-1} \|\theta^{N+1}\|^2 \\
 F^{11} &\leq \Theta_2 \tau^2 \|\nabla_{d,r} u^N\|^2 + 2\lambda h^{-1} \|\theta^{N+1}\|^2 \\
 F^{12} &\leq \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\theta^{N+1,0}\|^2 + \|\nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{13} &\leq \tau^2 \frac{\|b(\pi_h u^{N,0})\|^2}{2} \|\theta^{N,0}\|^2 + \|\nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{14} &\leq \Theta_3 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2, F^{15} \leq \Theta_4 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2 \\
 F^{16} &\leq \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16} \|\theta^{N+1,0}\|^2 + \|\theta^{N+1,0}\|^2 \\
 F^{17} &\leq \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16} \|\theta^{N+1,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{18} &\leq \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16} \|\theta^{N,0}\|^2 + \|\theta^{N+1,0}\|^2 \\
 F^{19} &\leq \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^N)\|^2}{16} \|\theta^{N,0}\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{20} &\leq \Theta_5 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2, F^{21} \leq \Theta_6 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 \\
 F^{22} &\leq \Theta_7 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \|\theta^{N+1,0}\|^2, F^{23} \leq \Theta_8 \tau^2 \|\nabla_{d,r} \theta^N\|^2 + \frac{\delta}{4} \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2
 \end{aligned}$$

Then the above equation becomes, we obtain;

$$4\lambda h^{-1} \|\theta^{N+1}\|^2 + \left[ \tau \frac{\|\pi_h a(u^{N,0})\|^2}{2} + 2 \right] \|\nabla_{d,r} \theta^{N+1}\|^2 \leq$$

$$\begin{aligned}
 & - \left[ \rho + 9 + \tau \|c\| + \tau^2 \frac{\|\pi_h b(u^{N,0})\|^2}{2} + \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16} \right] \|\theta^{N+1,0}\|^2 \\
 & - \left[ \frac{5}{4} + \frac{\tau^2}{16} \|c\|^2 + \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16} + \tau^2 \frac{\|\pi_h b(u^{N,0})\|^2}{2} \right. \\
 & \quad \left. + \tau^2 \frac{\|\pi_h \nabla_{d,r} b(u^N)\|^2}{16} \right] \|\theta^{N,0}\|^2 - 2\delta \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 - \\
 & \left[ \tau^2 \frac{\|a(\pi_h u^{N,0})\|^2}{16} + 1 + \mu \tau^2 \right] \|\nabla_{d,r} \theta^N\|^2 - \Theta_1 \tau^2 \|\pi_h \nabla_{d,r} u^{N+1}\|^2 - \Theta_2 \tau^2 \|\nabla_{d,r} u^N\|^2
 \end{aligned}$$

Then

$$\begin{aligned}
 \|\theta^{N+1,0}\|^2 + \|\nabla_{d,r} \theta^{N+1}\|^2 & \leq \eta_1 h [\tau^2 (\|\pi_h \nabla_{d,r} u^{N+1}\|^2 + \|\nabla_{d,r} u_h^N\|^2 + [1 + \tau + \tau^2] \|\theta^{N+1,0}\|^2 \\
 & + \tau^2 \|\theta^{N,0}\|^2 + \|\beta \cdot \nabla_{d,r} \theta^{N+1}\|^2 + \tau^2 \|\nabla_{d,r} \theta^{N,0}\|^2 + (1 + \tau^2) \|\theta^{N,0}\|^2], \quad \blacksquare
 \end{aligned}$$

Where,  $\eta_1 = \max\{-\Theta_1, -\Theta_2, -(\rho + 9), -\|c\|, -\frac{\|\pi_h b(u^{N,0})\|^2}{2}, -\frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16}$

$$, -\left[\frac{\|a(\pi_h u^{N,0})\|^2}{16} + \mu\right], -\frac{5}{4}, -\frac{\|c\|^2}{16}, -\frac{\|\pi_h \nabla_{d,r} b(u^{N,0})\|^2}{16}, -\frac{\|\pi_h b(u^{N,0})\|^2}{2}, \\
 -\frac{\|\pi_h \nabla_{d,r} b(u^N)\|^2}{16}, -2\delta\} / \min\{4\lambda, [\frac{\|\pi_h a(u^{N,0})\|^2}{2} + 3]\}$$

and  $\mu = \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7 + \Theta_8$

**8. The Numerical Results:** We offer the Petrov weak Galerkin finite element approach to show its effectiveness and dependability in tackling non-linear convection-diffusion situations. a numerical example. All computation are performed using MATLAB. The equations (1.1-1.3) are examined here (Saadoon, 2020).

$$\frac{\partial u}{\partial t} - \nabla \cdot (\epsilon \nabla u) + u \cdot \nabla u + cu = f, \quad \Omega \times (0, T], \quad (8.1)$$

$$u(x, t) = 0: \quad \Gamma \times (0, T], \quad (8.2)$$

$$u(x, 0) = u^0(x); \quad x \in \bar{\Omega}, \quad (8.3)$$

The uniform triangular partition is used in the example, and the spatial mesh size is indicated by ( $h = 1/2; 1/4; 1/8; 1/16; 1/32$ ). Calculate  $f$  and  $g$  by computed from exact solution for  $\Omega = [0, 1]^2$ .

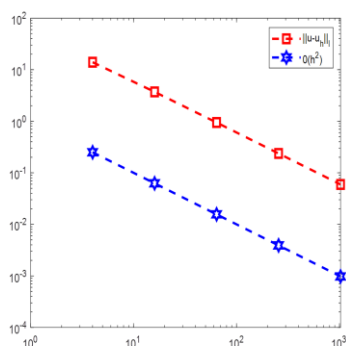
$$u(x, y, t) = e^{-1} x(1-x)y(1-y).$$

We used " $\epsilon = 0.0001, t = 0.2, \Delta t = 0.001$ " We determined the error value in table 1 and showed the order error scheme in fig. (1). Figures (2) give the PWG numerical solution and exact solution for the mesh size  $h = \frac{1}{32}$

Table 1:  $H^1 - error$  and  $L^2 - error$  with  $t = 0.2, \epsilon = 0.0001$  and,  $\Delta t = 0.001$

| H              | $H^1 - error$ | $H^1 - order$ | $L^2 - error$ | $L^2 - order$ |
|----------------|---------------|---------------|---------------|---------------|
| $\frac{1}{2}$  | 0.07601       | 0             | 0.08721       | 0             |
| $\frac{1}{4}$  | 0.03994       | 0.928354      | 0.021809      | 1.999569      |
| $\frac{1}{8}$  | 0.02123       | 0.911729      | 0.005491      | 1.989782      |
| $\frac{1}{16}$ | 0.01129       | 0.9110588     | 0.00138       | 1.992400      |
| $\frac{1}{32}$ | 0.000598      | 0.9168280     | 0.00034       | 2.001061      |

$H^1 - Order error$



$L^2 - Order error$

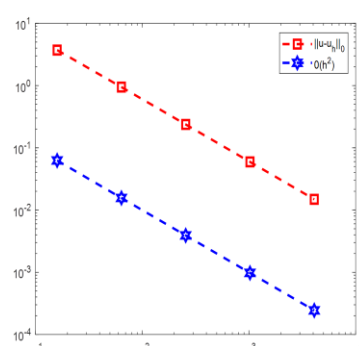


Figure 1. Order error for the velocity with  $H^1 - Order error$  and  $L^2 - Order error$

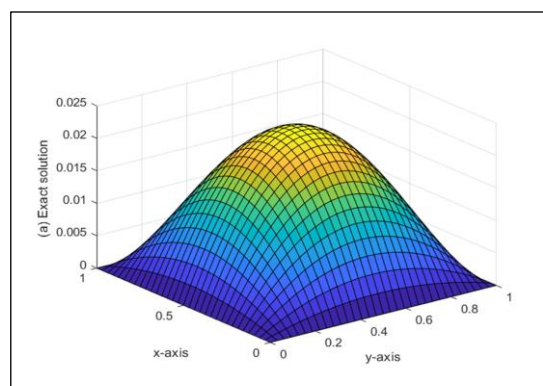
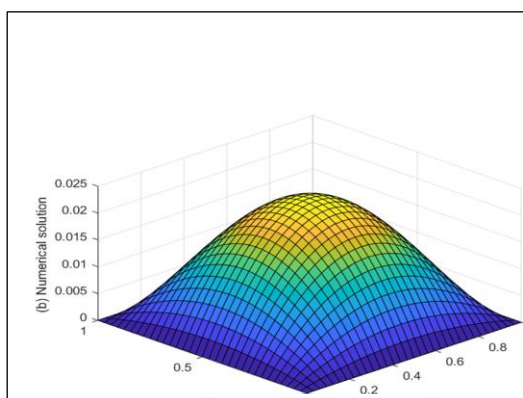


Fig 2. (a) PWG exact solution with  $\epsilon = 0.001$ . (b) PWG numerical solution with  $\epsilon = 0.001$ .

## 9. Conclusions:

In this paper, we provide the Petrov weak Galerkin finite element method, which includes the Crank-Nicolson strategy for solving a non-linear equation and the stabilizer of weak functions. issue of convection-diffusion. We have established that the error equation and convergence rates are optimal in the  $H^1$  and  $L^2$  norms after analyzing the stability. The theory has been validated by numerical examples, and the Petrov weak Galerkin method's flexibility and convergence lead to accurate approximations to the solution.

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