The Error Analysis for Linearized Crank-Nicolson-Galerkin Method for Navier-Stokes Problem

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Abstract:

In this paper, we studied the non-stationary incompressible Navier-Stokes problem in two-dimensional domain by using mixed finite element method. By using the Linearized Crank-Nicolson-Galerkin Method we found the weak form to the above problem which is then improved to the approximate solution. These estimates are then applied to obtain quasi-optimal error analysis in the energy norm for velocity, pressure and velocity with pressure.

Keyword: Navier-Stokes problem, Mixed finite element method, Linearized Crank-Nicolson-Galerkin Method, Error analysis.

تحليل الخطأ لمسألة نافيير- ستوكس بطريقة

كرانك-نيكلسون–كاليركين الخطية

مستخلص

في هذا البحث، درسنا مسألة نافيير- ستوكس غير المضغوطة غير الثابتة في مجال ثنائي البعد مستخدمين طريقة العنصر المنتهي المختلطة. بواسطة طريقة كرانك-نيكلسون–كاليركين الخطية وجدنا

الصيغة الضعيفة للمسألة أعلاه وثم حسبنا الحل التقريبي . تم تطبيق هذه التخمينات للحصول على الحالة شبه المثالية لتحليل الخطأ في معيار الطاقة للسرعة والضغط والسرعة مع الضغط.

1. Introduction

The classical numerical method for partial differential equations is the difference method where the discrete problem is obtained by replacing derivatives with difference quotients involving the values of the unknown at certain points.

The finite element method is a numerical analysis technique for obtaining approximate solutions to a wide variety of problems in mechanics and physics[5]. Although originally developed to study stresses in complex airframe structures, it has since been extended and applied to the broad field of continuum mechanics. Because of its diversity and flexibility as an analysis tool, it is receiving much attention in engineering schools and in industry. In this method, the discretization procedures reduce the problem to one of a finite number of unknowns by dividing the solution region into elements and by expressing the unknown field variable in terms of assumed approximating functions within each element. The approximating functions (sometimes called interpolation functions) are defined in terms of the values of the field variables at specified points called nodes or nodal points[9].

Mixed finite element methods are one of the important approaches for solving system of partial differential equations, for example, the stationary Navier-Stokes equations. However, fully discrete system of mixed finite element solutions for the stationary Navier-Stokes equations is of many degrees of freedom[8].

1.1 Notation

Let Ω be an open and bounded domain in R^2 with Lipschitz continuous boundary Γ .Throughout this paper we will use the standard notation for Sobolev spaces. Specially $H^r(\Omega)$, where r is an integer greater than zero, will denotes the Sobolev space of real-valued functions with square integrable derivatives of order up to r equipped with the usual norm which we denote $\|\cdot\|_r$. We will denote $H^0(\Omega)$by $L^2(\Omega)$ and the standard L^2 inner product by $(\cdot,\cdot)$. Also $H^r(\Omega)$will denote the space of vector-valued functions each of whose n components belong to $H^r(\Omega)$and the dual space of $H^r(\Omega)$will be denoted by $H^{-r}(\Omega)$.Of particular interest to us will be the constrained space see [9]

$$V = [H_0^1(\Omega)]^2 = \{v = (v_1, v_2) : v_i \in H_0^1, \quad i = 1,2\}$$

and

$$Q = \left\{ q \in L_2(\Omega) : \int_\Omega q \; dx = 0 \right\}$$

1.2 The weak formulations

We are interested in approximating the solution of the Navier–Stokes equations written in the primitive variable formulation of the velocity $u = (u_1, u_2)$ and the pressure p. In particular, we consider the steady Navier–Stokes equations, see [3].

$$\frac{\partial u}{\partial t} + u.\nabla u - e\Delta u + \nabla p = f \qquad in\ \Omega \qquad (1.1a)$$

$$\nabla \cdot u = 0 \qquad in\ \Omega \qquad (1.1b)$$

$$u = 0 \qquad on\ \partial\Omega = \Gamma \qquad (1.1c)$$

where and $f \in H^1$is given the body force per unit mass. In the following exposition represent to e is the inverse Reynolds number.

Multiplying (1.1a) and (1.1b) by $v \in V$ and $q \in Q$, respectively, as a test functions and take integral over Ω

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx - \int_\Omega e\Delta u\, v\,dx + \int_\Omega (\nabla u) u\, v\,dx + \int_\Omega \nabla p\, v\,dx = \int_\Omega f\ v\,dx; v \in V\,,$$

$$\int_\Omega div\, u\ q\,dx = 0\,; \quad q \in Q,$$

by using Green's formulation

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx + e\int_\Omega \nabla u\ \nabla v\,dx - \frac{1}{2}\int_\Omega u^2\ \nabla v\,dx - \int_\Omega p\ div\ v\,dx = \int_\Omega f\ v\,dx\ ,$$

$$\int_\Omega div\, u\ q\,dx = 0\,.$$

We consider the following standard weak formulation of non- steady: seek $(u, p) \in V \times Q$ such that

$$(u_t, v) + a(u, v) - n(u, u, v) - b(v, p) = (f, v); \quad v \in V\,, \tag{1.2a}$$

$$b(u, q) = 0\,; \quad q \in Q\,, \tag{1.2b}$$

where

$$a(u, v) = e\int_\Omega \nabla u\ \nabla v\ dx\,,$$

$$n(u, u, v) = \frac{1}{2}\int_\Omega u^2\ \nabla v\,dx\,,$$

$$b(u, q) = \int_\Omega div\, u\ q\ dx\,.$$

Continuity of the forms $a(\cdot,\cdot), n(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot)$ can be demonstrated. These conditions guarantee the existence and uniqueness of a solution (u, p) [3].

1.3 The Fully-Discrete Approximation

Now we turn our attention to some simple schemes for discretization with respect to the time variable.

1.3.1 Crank-Nicolson-Galerkin Method for Weak Formulations

Letting τ be the time step and u^n the solution in V of $u(\cdot,t_n), n=1,2,\cdots,N$, at $t=t_n=n\tau$. This method is defined by replacing the time derivative u_t in problem (1.2) by backward differences quotient $\frac{(u^n-u^{n-1})}{\tau}$ and the u and p by differences quotient $\frac{(u^n+u^{n-1})}{2}$ and $\frac{(p^n+p^{n-1})}{2}$ with the corresponding discretization error is $O(\tau^2)$

$$\left(\frac{u^n-u^{n-1}}{\tau},v\right)+a\left(\frac{u^n+u^{n-1}}{2},v\right)-n\left(\frac{u^n+u^{n-1}}{2},\frac{u^n+u^{n-1}}{2},v\right)-b\left(v,\frac{p^n+p^{n-1}}{2}\right)=\left(\frac{f(t_n)+f(t_{n-1})}{2},v\right) ,$$

$$;\forall v\in V,\ n=1,\cdots N \tag{1.3a}$$

$$b\left(\frac{u^n+u^{n-1}}{2},q\right)=0 \quad ;\forall q\in Q \tag{1.3b}$$

1.3.2 Linearized Crank-Nicolson-Galerkin Method for Weak Formulations

Problem (1.3) shares, however with backward Euler method discussed first above, the disadvantage of producing, at each time level, a nonlinear system of problem. For this reason we shall consider also a linearized modification in which the argument of $n(\cdot,\cdot,\cdot)$ is obtained by extrapolation from u^{n-1} and u^{n-2}, [2], with

$$\widehat{u}^{n}=\frac{3}{2}u^{n-1}-\frac{1}{2}u^{n-2}$$

$$\frac{1}{\tau}\left(u^{n}-u^{n-1},v\right)+a\left(\bar{u}^{n},v\right)-n\left(\widehat{u}^{n},\widehat{u}^{n},v\right)-b\left(v,\bar{p}^{n}\right)=\left(\bar{f}(t_{n}),v\right)\quad;\forall v\in V,\ n=1,\cdots,N \tag{1.4a}$$

$$b\left(\bar{u}^{n},q\right)=0\quad;\forall q\in Q \tag{1.4b}$$

where

$$\bar{u}^{n}=\frac{u^{n}+u^{n-1}}{2},\ \bar{p}^{n}=\frac{p^{n}+p^{n-1}}{2},\quad \bar{f}(t_{n})=\frac{f(t_{n})+f(t_{n-1})}{2}$$

1.3.3 Linearized Crank-Nicolson-Galerkin Method for Discrete Problem

Given finite dimensional spaces $V_h\subset V$ and $Q_h\subset Q$ where $0<h<1$ then the approximate solution (u_h,p_h) to (u,p) is the solution of the following problem:

$$\frac{1}{\tau}\left(u_h^{n}-u_h^{n-1},v\right)+a\left(\bar{u}_h^{n},v\right)-n\left(\widehat{u}_h^{n},\widehat{u}_h^{n},v\right)-b\left(v,\bar{p}_h^{n}\right)=\left(\bar{f}(t_{n}),v\right)\quad;\forall v\in V_h,\ n=1,\cdots,N \tag{1.5a}$$

$$b\left(\bar{u}_h^{n},q\right)=0\quad;\forall q\in Q_h \tag{1.5b}$$

The nonlinear equation (1.5a) will be solvable for u^n when u^{n-1} and u^{n-2} are given. Choosing $n(\cdot,\cdot,\cdot)$ at u^{n-1} as we did for the back ward Euler scheme will not be satisfactory here since this would be less accurate than necessary, whereas since

$$\widehat{u}^{n}=\frac{3}{2}u^{n-1}-\frac{1}{2}u^{n-2}=u^{n-\frac{1}{2}}+O\left(\tau^{2}\right)\quad as\quad \tau\to 0$$

the choice just proposed will have the desired accuracy.

2. Abstract Results

Let V and Q be two real Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)\in L^\infty$ be continuous bilinear forms on $V\times V$ and $V\times Q$ respectively [4], $n(\cdot,\cdot,\cdot)\in L^\infty$ be continuous trilinear form on $V\times V\times V$ [8]:

$$|a(u,v)|\leq\|a\|_{L^\infty}\cdot\|u\|_V\cdot\|v\|_V \qquad \forall\, u,v\in V, \tag{2.1}$$

$$|n(u,u,v)|\leq\|n\|_{L^\infty}\cdot\|u\|_V^2\cdot\|v\|_V \qquad \forall\, u,v\in V, \tag{2.2}$$

$$|b(u,p)|\leq\|b\|_{L^\infty}\cdot\|u\|_V\cdot\|p\|_Q \qquad \forall\, u\in V;\ \forall p\in Q. \tag{2.3}$$

we now state several further assumptions which we will require in the proofs of our main results [4].

(H1) There is a constant α >0 (α independent of h) such that

$$a(v,v)\geq\alpha\,\|v\|_W^2 \qquad \forall v\in Z_h\,,$$

where $$Z_h=\{v\in V_h : b(v,\varphi)=0\,,\ \forall\varphi\in Q_h\}$$

(H2) $S(h)$ is a number satisfying $\|v\|_V\leq S(h)\|v\|_W\,;\quad \forall v\in V_h\,.$

(H3) There is a linear operator $\Pi_h : Y\rightarrow V_h$ satisfying

$b(y-\Pi_h y,\varphi)=0\ ;\quad \forall y\in Y \quad and \quad \varphi\in Q_h\,.$

Definition 2.1 [6] Cauchy-Schwarz inequalities:

$$\left|(v,w)_{L^2(\Omega)}\right|\leq\|v\|_{L^2(\Omega)}\,\|w\|_{L^2(\Omega)}\,;\quad v,\,w\in L^2(\Omega), \tag{2.4}$$

and

$$\left|(v,w)_{H^1(\Omega)}\right|\leq\|v\|_{H^1(\Omega)}\,\|w\|_{H^1(\Omega)}\,;\quad v,\,w\in H^1(\Omega)\,. \tag{2.5}$$

Lemma 2.1 There exists a linear operator $\Pi_h : H \to H_h$ such that, [5]

$(div\,\Pi_h U\,, v_h) = (div\,U\,, v_h)\,; \quad \forall v_h \in V_h\,, \forall U \in H\,,$

$$\|\Pi_h U - U\| \le Ch^s \|U\|_s\,; \quad for\ s = 1, 2\,.$$

3. Error Estimated

We shall now study the errors $u^n - u_h^n$ and $p^n - p_h^n$ where u^n and p^n are the solution of weak form and u_h^n and p_h^n are the solution of the mixed finite element problem (V_h *and* Q_h)

Theorem 3.1 Let $u^n \in V$ be the solution of problem (1.4) and $u_h^n \in V_h$ is the approximation solution of problem (1.5). Then, there exists a constant $C > 0$ independent of h *and* τ such that:

$$\|u^n - u_h^n\| \le C(h^r + \tau). \tag{3.1}$$

Proof: Let $u^n - u_h^n = (u^n - \Pi_h u^n) - (u_h^n - \Pi_h u^n) = \rho^n - \theta^n$

For each time step n and each norm, we apply the triangle inequality

$\|u^n - u_h^n\| \le \|\rho^n\| + \|\theta^n\|$

from Lemma 2.1 $\qquad \|\rho^n\| \le Ch^r \|u^n\|,$

To find a bound on θ^n term, note that

$$\frac{1}{\tau}\left(\theta^n-\theta^{n-1},\varphi\right)+a\left(\bar{\theta}^n,\varphi\right)-n\left(\hat{\theta}^n,\hat{\theta}^n,\varphi\right)-b\left(\varphi,\bar{p}_h^n-\bar{p}^n\right)=$$
$$\frac{1}{\tau}\left(u_h^n-u_h^{n-1},\varphi\right)+a\left(\bar{u}_h^n,\varphi\right)-n\left(\hat{u}_h^n,\hat{u}_h^n,\varphi\right)-b\left(\varphi,\bar{p}_h^n\right)$$
$$-\frac{1}{\tau}\left(\Pi_h u^n-\Pi_h u^{n-1},\varphi\right)-a\left(\Pi_h\bar{u}^n,\varphi\right)+n\left(\Pi_h\hat{u}^n,\Pi_h\hat{u}^n,\varphi\right)+b\left(\varphi,\bar{p}^n\right).$$

By the definition of interpolation, we have

$$A\left(u^n-\Pi_h u^n,\varphi\right)=0,$$

also note that

$$\frac{1}{\tau}\left(u_h^n-u_h^{n-1},\varphi\right)+a\left(\bar{u}_h^n,\varphi\right)-n\left(\hat{u}_h^n,\hat{u}_h^n,\varphi\right)-b\left(\varphi,\bar{p}_h^n\right)=\left(\bar{f}^n,\varphi\right),$$

then, we have

$$\frac{1}{\tau}\left(\theta^n-\theta^{n-1},\varphi\right)+a\left(\bar{\theta}^n,\varphi\right)-n\left(\hat{\theta}^n,\hat{\theta}^n,\varphi\right)-b\left(\varphi,\bar{p}_h^n-\bar{p}^n\right)=\left(\bar{f}^n,\varphi\right)$$
$$-\frac{1}{\tau}\left(\Pi_h u^n-\Pi_h u^{n-1},\varphi\right)+a\left(\Pi_h\bar{u}^n,\varphi\right)-n\left(\Pi_h\hat{u}^n,\Pi_h\hat{u}^n,\varphi\right)-b\left(\varphi,\bar{p}^n\right),$$

$$=\left(u_t^n,\varphi\right)-\frac{1}{\tau}\left(\Pi_h u^n-\Pi_h u^{n-1},\varphi\right).$$

Adding and subtracting $\frac{1}{\tau}\left(u^n-u^{n-1},\varphi\right)$ gives,

$$\frac{1}{\tau}\left(\theta^n-\theta^{n-1},\varphi\right)+a\left(\bar{\theta}^n,\varphi\right)-n\left(\hat{\theta}^n,\hat{\theta}^n,\varphi\right)-b\left(\varphi,\bar{p}_h^n-\bar{p}^n\right)=$$
$$\frac{1}{\tau}\left(u^n-u^{n-1},\varphi\right)-\frac{1}{\tau}\left(\Pi_h u^n-\Pi_h u^{n-1},\varphi\right)+\left(u_t^n,\varphi\right)-\frac{1}{\tau}\left(u^n-u^{n-1},\varphi\right)$$

$$=\frac{1}{\tau}\left(\rho^n-\rho^{n-1},\varphi\right)+\left(\xi^n,\varphi\right),$$

where

$$\xi^n = u_t^n - \frac{1}{\tau}\left(u^n - u^{n-1}\right)$$

choosing $\varphi = \bar{\theta}^n$ and $\bar{p}_h^n = q^n$

$$\frac{1}{\tau}\left(\theta^n - \theta^{n-1}, \bar{\theta}^n\right) + a\left(\bar{\theta}^n, \bar{\theta}^n\right) - n\left(\hat{\theta}^n, \hat{\theta}^n, \bar{\theta}^n\right) =$$
$$\frac{1}{\tau}\left(\rho^n - \rho^{n-1}, \bar{\theta}^n\right) + \left(\xi^n, \bar{\theta}^n\right) - b\left(\bar{\theta}^n, \bar{p}^n - q^n\right).$$

By using (2.1), (2.2) and (2.3), and multiplying by τ, we get

$$\|\theta^n\|\|\bar{\theta}^n\| + \alpha\tau\|\bar{\theta}^n\|^2 + \beta\tau\|\hat{\theta}^n\|^2\|\bar{\theta}^n\| \le \|\theta^{n-1}\|\|\bar{\theta}^n\| + \|\rho^n - \rho^{n-1}\|\|\bar{\theta}^n\| + \tau\|\xi^n\|\|\bar{\theta}^n\| + S(h)\tau\|\bar{\theta}^n\|\|\bar{p}^n - q^n\|, \tag{3.2}$$

applying Young's inequality two sides gives,

$$\frac{1}{2}\|\theta^n\|^2 + \frac{1}{2}\|\bar{\theta}^n\|^2 + \alpha\tau\|\bar{\theta}^n\|^2 + \beta\tau\left[\frac{1}{4}\|\hat{\theta}^n\|^4 + \|\bar{\theta}^n\|^2\right] \le \frac{1}{2}\|\theta^{n-1}\|^2 + \frac{1}{2}\|\bar{\theta}^n\|^2 +$$
$$\frac{1}{4\alpha\tau}\|\rho^n - \rho^{n-1}\|^2 + \alpha\tau\|\bar{\theta}^n\|^2 + \tau\left[\frac{1}{2}\|\xi^n\|^2 + \frac{1}{2}\|\bar{\theta}^n\|^2\right] +$$
$$S(h)\tau\left[\frac{1}{2}\|\bar{p}^n - q^n\|^2 + \frac{1}{2}\|\bar{\theta}^n\|^2\right],$$

choosing $S(h) = 1 = \beta$ **,** and multiplying 2 and rearranging gives

$$\|\theta^n\|^2 + \frac{\tau}{2}\|\hat{\theta}^n\|^4 \le \|\theta^{n-1}\|^2 + \frac{1}{2\alpha\tau}\|\rho^n - \rho^{n-1}\|^2 + \tau\|\xi^n\|^2 + \tau\|\bar{p}^n - q^n\|^2,$$

since, $\frac{\tau}{2}\|\hat{\theta}^n\|^4 \ge 0$,

then,

$$\|\theta^n\|^2 \le \|\theta^{n-1}\|^2 + \frac{1}{2\alpha\tau}\|\rho^n - \rho^{n-1}\|^2 + \tau\|\xi^n\|^2 + \tau\|\bar{p}^n - q^n\|^2,$$

Summing both sides from $n=1$ to $n=N$, we get

$$\left\|\theta^N\right\|^2 \le \left\|\theta^0\right\|^2 + \frac{1}{2\alpha\tau}\sum_{n=1}^{N}\left\|\rho^n-\rho^{n-1}\right\|^2 + \tau\sum_{n=1}^{N}\left\|\xi^n\right\|^2 + \tau\sum_{n=1}^{N}\left\|\bar{p}^n-q^n\right\|^2 , \qquad (3.3)$$

For the second term note that

$$\rho^n-\rho^{n-1} = \int_{t_{n-1}}^{t_n}\rho_t\, dt \ ,$$

this implies

$$\left\|\rho^n-\rho^{n-1}\right\| = \int_{t_{n-1}}^{t_n}\left\|\rho_t\right\| dt \ ,$$

thus

$$\left\|\rho^n-\rho^{n-1}\right\|^2 \le \left(\int_{t_{n-1}}^{t_n}\left\|\rho_t\right\| dt\right)^2 = \tau^2\left(\int_{t_{n-1}}^{t_n}\left\|\rho_t\right\|\frac{dt}{\tau}\right)^2 \ ,$$

applying Jensen's inequality (see [1]) to the right hand side

$$\left\|\rho^n-\rho^{n-1}\right\|^2 \le \tau^2\int_{t_{n-1}}^{t_n}\left\|\rho_t\right\|^2\frac{dt}{\tau} = \tau\int_{t_{n-1}}^{t_n}\left\|\rho_t\right\|^2 dt \ ,$$

this implies

$$\frac{1}{\tau}\sum_{n=1}^{N}\left\|\rho^n-\rho^{n-1}\right\|^2 \le \int_0^T\left\|\rho_t\right\|^2 dt \le C_1 h_u^{2r}\int_0^T\left\|u_t\right\|_r^2 dt = C_1 h_u^{2r}\left\|u_t\right\|_r \ , \qquad (3.4)$$

$$\frac{1}{2\alpha\tau}\sum_{n=1}^{N}\left\|\rho^n-\rho^{n-1}\right\|^2 \le C_2 h_u^{2r}\left\|u_t\right\|_r \ .$$

To bound the third term of (3.3), note that

$$\xi^n = u_t^n - \frac{1}{\tau}\left(u^n - u^{n-1}\right) \quad ,$$

then, $\tau\,\xi^n = \tau\,u_t^n - \int_{t_{n-1}}^{t_n} u_t\,dt = \left(t_n - t_{n-1}\right)u_t^n - \int_{t_{n-1}}^{t_n} u_t\,dt$,

from [Theorem 3.5.1 (p 38) in [1]], we get $\|\xi^n\| = \int_{t_{n-1}}^{t_n} \|u_{tt}\|\,dt$,

and too $\tau \sum_{n=1}^{N} \|\xi^n\|^2 \le \tau \|u_{tt}\|_{L^2}^2$.

To bound the fourth term of (3.3), note that

$$\|\bar{p}^n - q^n\|^2 \le C_3\,h_p^{2r} \,.$$

Applying these results to (3.3) gives,

$$\|\theta^N\|^2 \le \left(\|u_h^0 - u^0\| + C_4\,h_u^r\,\|u^0\|\right)^2 + C_2\,h_u^{2r}\,\|u_t\|_r + \tau\|u_{tt}\|_r^2 + \tau\sum_{n=1}^{N} C_3\,h_p^{2r} \quad ,$$

suppose that $h_u = h_p = h$ in this paper, this implies we get

$$\|\theta^N\| \le \|u_h^0 - u^0\| + C_5\left[h^r\left\{\|u^0\|_r + \|u_t\|_r\right\} + \tau\left\{\|u_{tt}\|_r + \mu\right\}\right], \qquad (3.5)$$

hence, $\|u^n - u_h^n\| \le C\left(h^r + \tau\right).$

The proof is complete

Theorem 3.2 Let $p^n \in Q$ be the solution of problem (1.4) and $p_h^n \in Q_h$ is the approximation solution of problem (1.5) then there exists a constant $C_6 > 0$ independent of h *and* τ such that:

$$\| \bar{p}^n - \bar{p}_h^n \| \le C_6 \left(h^r + \tau \right) \quad . \tag{3.6}$$

Proof: Put $v = \bar{u}_h^n$, $v = \bar{u}^n$ in equations (1.4a) , (1.5a) respectively, then subtracting the equations we find

$$\frac{1}{\tau}\left(\left(u^n - u_h^n\right) - \left(u^{n-1} - u_h^{n-1}\right), \bar{u}_h^n - \bar{u}^n\right) - n\left(\hat{u}^n - \hat{u}_h^n, \hat{u}^n - \hat{u}_h^n, \bar{u}_h^n - \bar{u}^n\right) - b\left(\bar{u}_h^n - \bar{u}^n, \bar{p}^n - \bar{p}_h^n\right) = \left(\bar{f}^n, \bar{u}_h^n - \bar{u}^n\right). \tag{3.7}$$

Let $\bar{p}^n - \bar{p}_h^n = \left(\bar{p}^n - \Pi_h \bar{p}^n\right) - \left(\bar{p}_h^n - \Pi_h \bar{p}^n\right) = \psi^n - \chi^n$,

by using triangle inequality, we have

$$\| \bar{p}^n - \bar{p}_h^n \| \le \|\psi^n\| + \|\chi^n\| \quad ,$$

from Lemma 2.1 $\|\psi^n\| \le C h^r \|\bar{p}^n\|$.

To estimate χ^n from equation (3.7), put $u^n - u_h^n = \rho^n - \theta^n$ and $\bar{p}^n - \bar{p}_h^n = \psi^n - \chi^n$

$$\frac{1}{\tau}\left(\left(\rho^n - \theta^n\right) - \left(\rho^{n-1} - \theta^{n-1}\right), \bar{\theta}^n - \bar{\rho}^n\right) - n\left(\hat{\rho}^n - \hat{\theta}^n, \hat{\rho}^n - \hat{\theta}^n, \bar{\theta}^n - \bar{\rho}^n\right) - b\left(\bar{\theta}^n - \bar{\rho}^n, \psi^n - \chi^n\right) = \left(\bar{f}^n, \bar{\theta}^n - \bar{\rho}^n\right) \quad ,$$

by using the elliptic projection , we get

$$\frac{1}{\tau}\left(\theta^n - \theta^{n-1}, \bar{\theta}^n\right) - n\left(\hat{\theta}^n, \hat{\theta}^n, \bar{\theta}^n\right) - b\left(\bar{\theta}^n, \chi^n\right) = \frac{1}{\tau}\left(\rho^n - \rho^{n-1}, \theta^n\right) - \left(\bar{f}^n, \bar{\theta}^n\right),$$

by using (2.1), (2.2) and (2.3), and multiplying by τ , we get

$$\|\theta^n\|\|\bar{\theta}^n\| + \beta\tau\|\hat{\theta}^n\|^2\|\bar{\theta}^n\| + S(h)\,\tau\|\bar{\theta}^n\|\|\chi^n\| \le \|\theta^{n-1}\|\|\bar{\theta}^n\| + \|\rho^n - \rho^{n-1}\|\|\bar{\theta}^n\| + \tau\|\bar{f}^n\|\|\bar{\theta}^n\|,$$

dividing by $\|\bar{\theta}^n\|$, we get

$$\left\|\theta^n\right\|+\beta\tau\left\|\hat{\theta}^n\right\|^2+S(h)\,\tau\left\|\chi^n\right\| \le \left\|\theta^{n-1}\right\|+\left\|\rho^n-\rho^{n-1}\right\|+\tau\left\|\bar{f}^n\right\| ,$$

since $\beta\tau\left\|\hat{\theta}^n\right\|^2 \ge 0$, we get

$$\left\|\chi^n\right\| \le \frac{1}{S(h)\tau}\left[\left\|\theta^{n-1}\right\|-\left\|\theta^n\right\|+\left\|\rho^n-\rho^{n-1}\right\|+\tau\left\|\bar{f}^n\right\|\right].$$

Summing both sides from $n=1$ to $n=N$, we have

$$\sum_{n=1}^{N}\left\|\chi^n\right\| \le \frac{1}{S(h)\tau}\left[\left\|\theta^0\right\|-\left\|\theta^N\right\|+\sum_{n=1}^{N}\left\|\rho^n-\rho^{n-1}\right\|+\tau\sum_{n=1}^{N}\left\|\bar{f}^n\right\|\right],$$

from equations (3.4) and (3.5), let $1\le n^*\le N$, we get

$$\left\|\chi^{n^*}\right\| \le \frac{1}{S(h)}\left[C\left(h^r+\tau\right)+C_1 h^r\left\|u_t\right\|+\left\|\bar{f}^n\right\|\right].$$

Hence, $\left\|\chi^{n^*}\right\| \le C_7\left(h^r+\tau\right).$

The proof is complete. □

Theorem 3.3 Let $\left(u^n, p^n\right)\in V\times Q$ is the solution of problem (1.4) and $\left(u_h^n, p_h^n\right)\in V_h\times Q_h$ is the approximation solution of problem (1.5), then, there exists a constant $C_8>0$ independent of h and τ such that:

$$\left\|u^n-u_h^n\right\|+\left\|\bar{p}^n-\bar{p}_h^n\right\|\le C_8\left(h^r+\tau\right). \tag{3.8}$$

Proof: We can prove this theorem from equations (3.1) and (3.6). □

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