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Compare Robust Wilk's statistics Based on MM-estimator for the Multivariate Multiple Linear Regression

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Abstract:

In the realm of multivariate linear regression, the classical Wilks' statistic stands out as a widely employed method for hypothesis testing, yet it exhibits high sensitivity to the influence of outliers. Numerous authors have explored non-robust test statistics grounded in normal theories across diverse scenarios. In this investigation, we developed a robust variant of the Wilks' statistics, utilizing the MM-estimator. This approach relies on observation weights determined through Hampel and Huber weight functions. We conducted a comparative analysis between the proposed statistics and the conventional Wilks' statistic. Monte Carlo studies were employed to assess the performance of the test statistics across various datasets, particularly under normal distribution conditions. The study delves into the comparative effectiveness of two test statistics—classical Wilks' and the newly proposed robust statistics. Both exhibited type I error rates and test power close to expected significance levels. However, in scenarios involving data contamination, the proposed statistical method demonstrated superior performance. It emerged as the preferred approach when dealing with corrupted or affected data.

Keywords: MM-Estimator, Outliers, Robustness, P-Value, Wilk's Statistic.

1. Introduction:

Let's suppose we have a q-variate dependent (predictor) vector $\boldsymbol{X_i} = \left(x_{i1}, x_{i2}, \dots, x_{iq}\right)^T)$ and a p-variate independent (response) vector $\boldsymbol{Y_i} = \left(y_{i1}, y_{i2}, \dots, y_{ip}\right)$. The model for multivariate multiple linear regression is expressed as follows:

$$Y = XB + \Xi \qquad (1)$$

Where $Y = (\boldsymbol{Y_1}, \dots, \boldsymbol{Y_n})^T, X = (\boldsymbol{1_n}, (\boldsymbol{X_1}, \dots, \boldsymbol{X_n})^T), \boldsymbol{1_n}$ is a n-dimensional vector whose all entries are 1, B is $((q \times 1) \times p)$ slope matrix and Ξ is $(p \times n)$ errors matrix. Multivariate regression finds practical applications in diverse fields such as engineering, biology, psychology, finance, and many others. Recent research studies on multivariate regression include works by (Friedman and Breiman,1997), (McKean and Davis,1993), (Ollia and

Koivunen,2003). To test the null hypothesis H_0 that there is no significant relationship between the set of dependent variables $\boldsymbol{Y}$ and the set of independent variables $\boldsymbol{X}$, meaning all population regression coefficients are zero, various statistics have been employed. The most widely used is Wilks' statistic Λ, defined as:

$$\Lambda = \frac{|E|}{|E+H|} \tag{2}$$

Where, E and H are given by:

$$E = Y^T Y - \hat{B} X^T Y , \tag{3}$$

$$H = \hat{B} X^T Y - n\bar{y}\bar{y}^T, \tag{4}$$

where

$$\hat{B} = (X^T X)^{-1} X^T Y \tag{5}$$

In the scenario where $\Lambda \leq \Lambda_{\alpha,p,v_E,v_H}$, the null hypothesis H_0 is rejected. Here, $\Lambda_{\alpha,p,v_E,v_H}$ represents the critical values with degrees of freedom $p,\ v_E = n - q - 1$ and $v_H = q$, at a significance level α in Wilks' critical values table. A significant Wilks' statistic, where its associated p-value is below a predetermined significance threshold α, provides evidence to reject the null hypothesis. This implies that, collectively, there is at least one independent variable that significantly influences the set of dependent variables. Assuming that Y follows a multivariate normal distribution, classical statistics are highly sensitive to the impact of outliers (Moller,2005). Several robust estimators of location and scatter in multivariate data, designed to withstand the influence of potential outliers, have been introduced. These include the M-estimator proposed by (Maronna ,1976), the Minimum Covariance Determinant Estimator (MCD) by (Rousseeuw ,1984), and the S-estimator by (Davies ,1987), (Rousseeuw and Leroy,1988), (Lopuhaa and Hendrik, 1989). In high dimensions, (Woodruff and Rocke,1994) explored a robust estimator of location and scatter. The simulation study in section 4 will discuss the impact of outliers on the Wilks' statistic. Therefore, we introduce an alternative robust Wilks' statistic in comparison to the classical Wilks' statistic. The MM-estimator (MM), introduced by (Yohai,1987), is utilized for its robustness in estimating scatter and location matrices. To enhance efficiency while maintaining robustness, we propose re-weighted steps for the MM estimator, as summarized in section 2. Accuracy assessments of the proposed approximations are presented in section 3. Section 4 employs a simulation study to evaluate the performance of the proposed statistics and compare different test statistics in various cases, considering factors such as robustness, test power, and significance level.

2. Robust Estimators:

Estimating the multivariate parameters of the dataset is a prerequisite for constructing robust Wilks' statistics. The MM-estimator proposed by (Yohai,1987) is recognized as an exceptionally robust estimator for both the multivariate scatter matrix and the location matrix. MM-estimator is established through a three-stage process. Initially, a preliminary regression estimate is calculated. This initial estimate is designed to be consistent, robust and possess a high breakdown point in the second phase, an M-estimate of the scale of errors is calculated, utilizing residuals derived from the initial regression estimate. M-estimates are a type of robust estimator that minimizes a certain objective function, often based on a robust function called a psi-function. This step aims to estimate the scale of the errors, providing information about the variability in the data. Lastly, in the third

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phase, an M-estimate of the regression parameters is computed. This is done using a proper redescending psi-function, which is a type of robust function that assigns less weight to extreme residuals. Redescending psi-functions are crucial for maintaining robustness in the presence of outliers. The three-stage procedure ensures that the estimator is robust at each step of the parameter estimation process. Effective algorithms for computing MM estimates are available in widely-used programming languages such as R, Python, SAS, and MATLAB.

3. The Proposed Wilks' Statistics:

Due to the complicity of the classical Wilks' distribution which was introduced by (Parvin,1958), we will use Bartlett approach for the Wilks' statistics distribution which is defined by (Rencher, 2002):

$$-\left(v_E-\frac{1}{2}\left(p-v_H+1\right)\right)\ln(\Lambda)\simeq\chi^2_{pv_H} \tag{6}$$

(Todorov and Filzmoser, 2010) presented an alternative re-weighted Wilks' statistic which defined as:

$$\Lambda_R=\frac{|\,E_R\,|}{|\,E_R+H_R\,|}\,, \tag{7}$$

where H_R and E_R in our proposed approximation are given by:

$$H_R=Y^T\left(WX(X^TWX)^{-1}X^TW-\frac{J_W}{\sum^n{}_0\,w_i}\right)Y\,, \tag{8}$$

$$E_R=Y^T\;(W-WX\;(X^TWX)^{-1}X^TW\;)\;Y\,, \tag{9}$$

where $W=diag(w_i),\quad i=1,2,\ldots,n,$

$$J_w=w^Tw,\qquad w=(w_1,w_2,\ldots,w_n)^T.$$

In our present study, we propose the following recommendations:

1. Find the estimated location vector $\hat{\mu}$ and the estimated covariance matrix $\hat{\Sigma}$ by using the MM-estimate.

2. Calculate the weights w_i of the observation $\boldsymbol{Y_i}$ by using the Hampel weight function (Campbell, 1980) and Huber weight function those defined as:

$$w_i=\begin{cases}1, & MD(\boldsymbol{Y_i})\le d_0\\ \frac{d}{MD(\boldsymbol{Y_i})}\,, & MD(\boldsymbol{Y_i})>d_0,\end{cases} \tag{10}$$

where

$$d=d_0e^{-\frac{1}{2}\left(\frac{MD(Y_i)-d_0}{b_2}\right)},\;\; d_0=\sqrt{p}+\frac{b_1}{\sqrt{2}},\quad b_1=2,\;\; b_2=1.25,$$

and $MD(Y_i)$ is the Mahalanobis distances which given by:

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$$MD(Y_i) = \sqrt{(\boldsymbol{Y_i} - \hat{\mu}^{\boldsymbol{0}})^{\boldsymbol{T}} \left(\hat{\Sigma}^0\right)^{-1} (\boldsymbol{Y_i} - \hat{\mu}^0)} \quad (11)$$

And w_i with Huber weight function is given by:

$$w_i = \begin{cases} 1, & MD(\boldsymbol{Y_i}) \le \sqrt{\chi^2_{0.975}(p)}, \\ 0, & wtherwise \end{cases} \quad (12)$$

3. For $j = j + 1$ calculate the weighted estimated location matrix $\hat{\mu}$ and the weighted estimated covariance matrix $\hat{\Sigma}$ as following:

$$\hat{\mu}^j = \frac{1}{\sum_1^n w_i} \sum_1^n w_i \boldsymbol{Y_i} \quad (13)$$

$$\hat{\Sigma}^j = \frac{1}{\sum_1^n w_i - 1} \sum_1^n w_i \left(\boldsymbol{Y_i} - \hat{\mu}^j\right)\left(\boldsymbol{Y_i} - \hat{\mu}^j\right)^T \quad (14)$$

4. Iterate through steps two and three until the following criterion is met:

$$\frac{\left|\hat{\Sigma}^{j+1}/p\right|^p}{det\left(\hat{\Sigma}^{j+1}\right)} \le \frac{\left|\hat{\Sigma}^{j}/p\right|^p}{det\left(\hat{\Sigma}^{j}\right)}$$

Now, we will introduce robust versions of Wilks' statistics, denoted as of Λ_{R1} and Λ_{R2}, similar to Λ_R in (7). These statistics rely on MM-estimators with Huber and Hampel weight functions, and we will construct their approximate distribution.

$$-\left(v_{E_R} - \frac{1}{2}\left(p - v_{H_R} + 1\right)\right) ln\Lambda_{R_W} \simeq \chi^2_p v_{H_R} \quad (15)$$

The degrees of freedom for v_{E_R} , and v_{H_R} the robust Wilks' statistic Λ_R can be found as follows:

$$v_{H_R} = trace\left(WX(X^TWX)^{-1} X^TW - \frac{J_W}{\sum_1^n w_i}\right) \quad (16)$$

$$v_{E_R} = trace(W - WX(X^TWX)^{-1} X^TW) \quad (17)$$

Now, let's utilize the QQ-plots technique to assess the accuracy of Λ_{R1} and Λ_{R2} through simulations involving k=3000 samples from the multivariate normal distribution. We will consider various scenarios, including different values for p (number of dependent variables), q (number of independent variables), and n (sample size). The accuracy of the classical distribution of the k statistics will be compared to the approximate distribution of Λ_{R1} and Λ_{R2} using QQ-plots. Some of these plots will be included in (Fig.1). The standard cut-off values for a test, 95%, 97.5%, and 99%, are presented in the plots as vertical lines. The plots, give an evident that the approximations are accurate across all dimensions considered in the study $p \in 2,3,4$ and $q \in 2,3,5$, for both large and small sample sizes, and across varying correlation levels r and $r = 0$.

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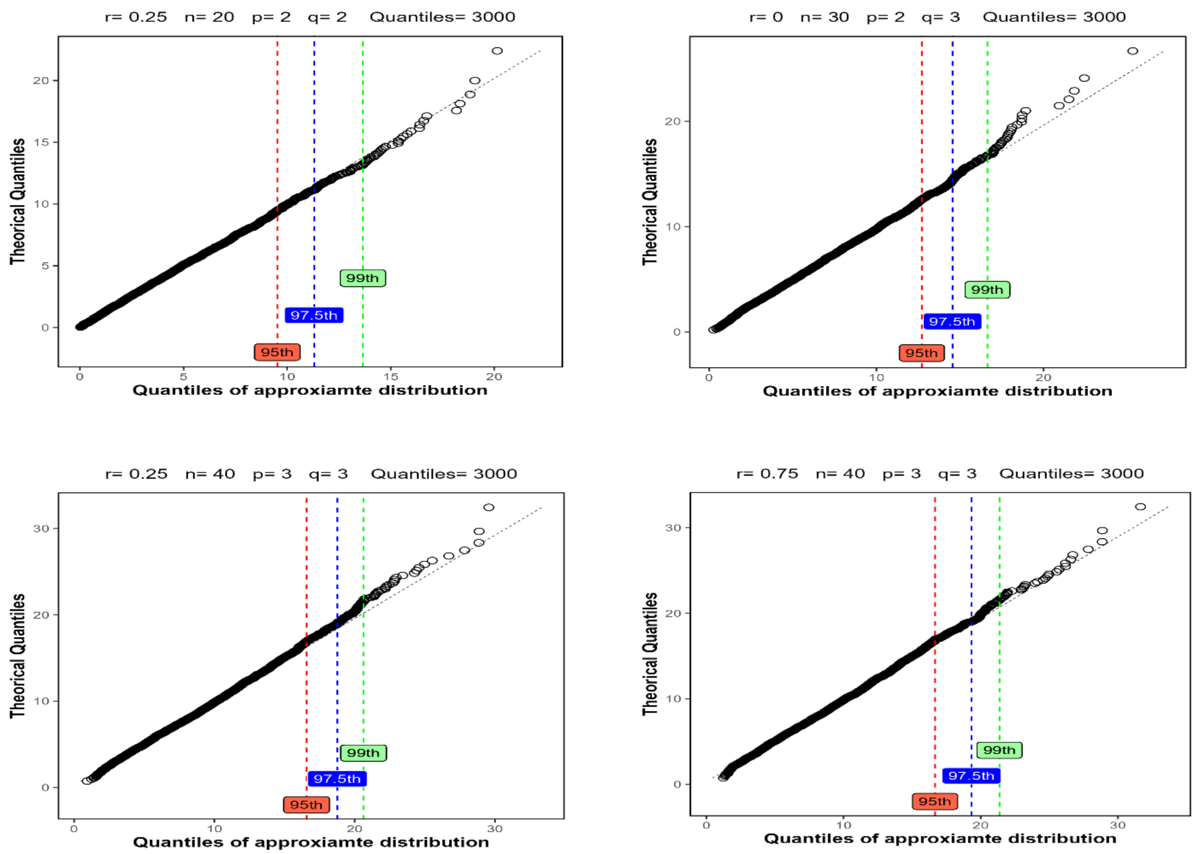

Figure 1. Λ_{R_1} ***and*** Λ_{R_2} QQ-plots

4.Monte Carlo Simulation:

The Monte Carlo process proves to be a valuable technique for evaluating the effectiveness of test statistics. We will assess the performance of these statistics based on both the type I error rate and the power of the test. By employing these measures, we will compare the behavior of robust statistics with the classical Wilks' statistic. This comparison will be conducted under two scenarios: one where the data includes outliers and another where the data follows a completely normal distribution. To study the type I error rate and the power of the test, we will consider various cases, including the number of independent variables $p = 2, 3, 4$, dependent variables $q = 2, 3, 5$, sample sizes $n = 20, 30, 40$, and correlations between components of the dependent variable Y. These correlations will be explored in terms of no correlation $r = 0$, medium correlation $r = 0.5$, and strong correlation $r = 0.75$.

4.1 Significance Level:

To assess and compare the rates of type I error for the test statistics under consideration, we generate observations of varying sizes n from both a multivariate normal distribution $\boldsymbol{Y_i} \sim N_p(\mathbf{0}, \boldsymbol{I})$ and a distribution containing outliers using the following model:

$$\boldsymbol{Y_s} \sim N_p(\mathbf{0}, \mathbf{I}), \qquad s = 1,2, \ldots, [m],$$

$$\boldsymbol{Y_t} \sim N_p(\mu^*, c\boldsymbol{I}), \qquad t = [m] + 1, \ldots, n$$

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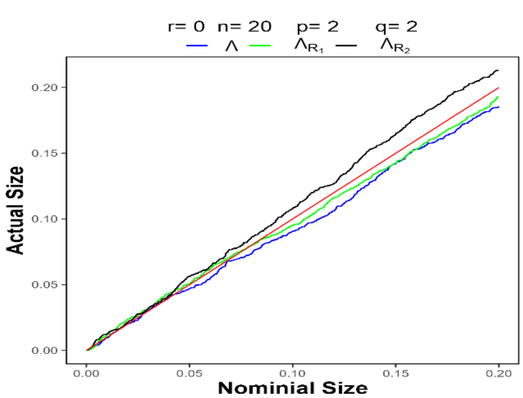
where $m = \frac{80n}{100}$, $\lceil m \rceil$ is the largest positive integer that is not less than m, $\mu^* = v^2 \sqrt{\chi^2_{p,0.001}}\, \mathbf{1}_p^T$, $v = 5, c = 0.0625$ and $\mathbf{1}_p$ is a vector whose all entries are 1. This is done under the null hypothesis $H_0 : B_1 = \mathbf{O}$, where B_1 is a matrix consisting of all rows of the coefficient matrix B except the first row. The classical Wilks' statistic Λ is compared to Bartlett's χ^2 formula in equation (7), and the robust Wilks' statistics proposed in this study are compared to the approximate distribution (Huber and Hampel) outlined in section 3. This simulation process is repeated $k = 3000$ times to calculate $\hat{\alpha} = \frac{T(k)}{k}$ (where $T(k)$, is the number of times the null hypothesis is rejected when it is true). The values of $\hat{\alpha}$ serve as estimates of the threshold significance level when the simulated critical values exceed the chosen significance level of 0.05. The nominal level interval is obtained from Fawcett and Salter's standard error formula (Salter and Facucett,1989). $\alpha \pm 2 \times \sqrt{\frac{\alpha(1-\alpha)}{k}}$ providing a standard deviation interval around the nominal level. We adopt P-value plots proposed by (Davidson and McKinnon,1998), as they offer a comprehensive representation of how the test statistics adhere to the approximate distribution under the null hypothesis across simulated samples. (Fig.2) depict the P-value plots, showing that the statistics Λ , Λ_{R_1} and Λ_{R_2} closely align with the 45° line.

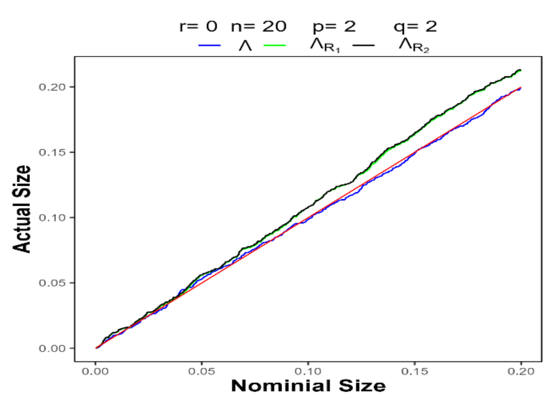
4.2 Power of the test:

To assess the power of the test for the considered statistics and make comparisons between them, we generate observations of varying sizes n from both a multivariate normal distribution $\boldsymbol{Y_i} \sim N_p(\mathbf{0}, \boldsymbol{I})$ and a distribution containing outliers using the model:

$$\boldsymbol{Y_s} \sim N_p(\mathbf{0}, \mathbf{I}), \qquad s = 1,2,\dots,\lceil m \rceil,$$
$$\boldsymbol{Y_t} \sim N_p(\mu^*, c\boldsymbol{I}), \qquad t = \lceil m \rceil + 1,\dots,n$$

where $m = \frac{80n}{100}$, $\lceil m \rceil$ is the largest positive integer that is not less than m, $\mu^* = v^2 \sqrt{\chi^2_{p,0.001}}\, \mathbf{1}_p^T$, $v = 5, c = 0.0625$ and $\mathbf{1}_p$ is a vector whose all entries are 1. This is done under the alternative hypothesis. We use Davidson and MacKinnon's method to compare the resulting power-size curves. (Fig.3) illustrates that the statistics Λ , Λ_{R_1} and Λ_{R_2} closely align with each other in terms of power size, considering normally distributed datasets without outliers. In cases where the data is corrupted and exhibits no or low correlation between dependent variables, Λ_{R_1} and Λ_{R_2} outperform Λ. However, in scenarios of high correlation, Λ , Λ_{R_1} and Λ_{R_2} show similar performance, as depicted in (Fig.4 and Fig.5).





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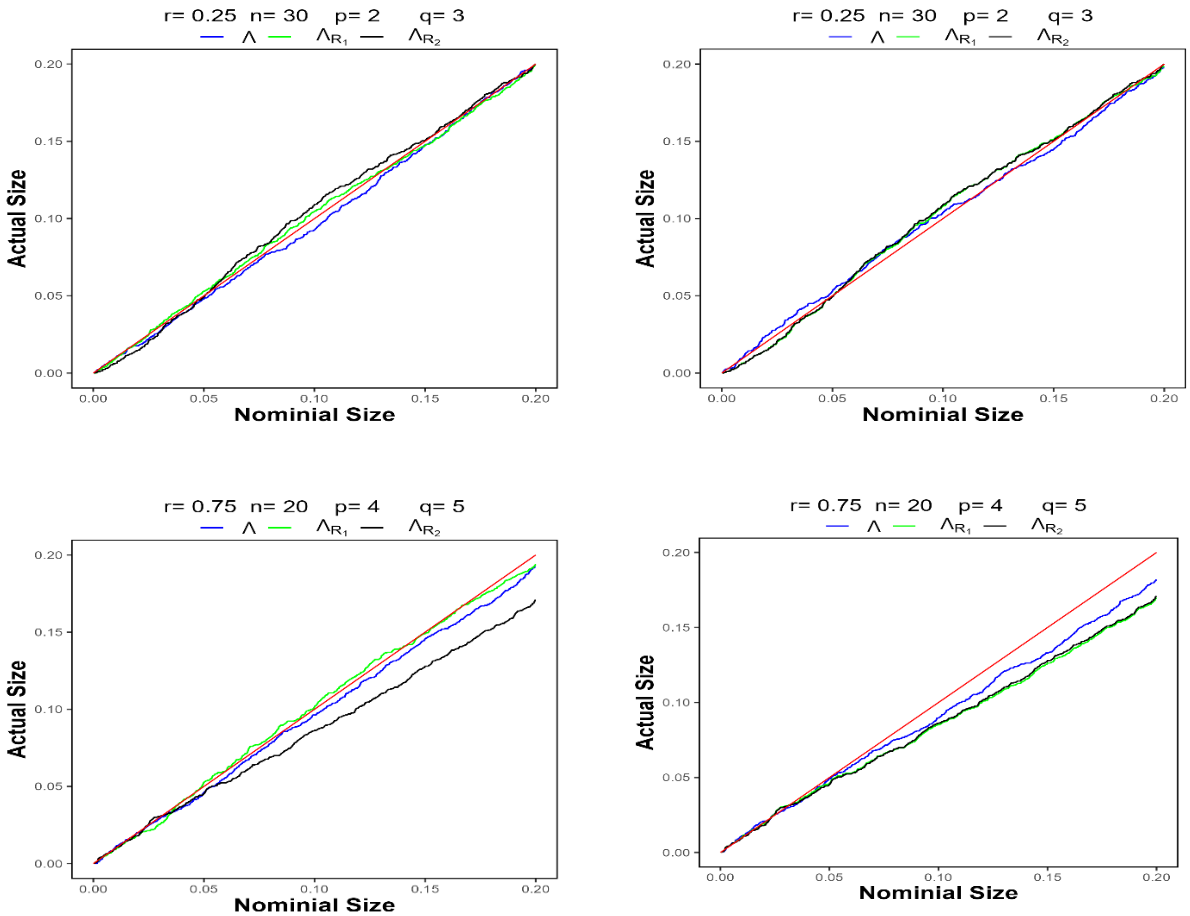
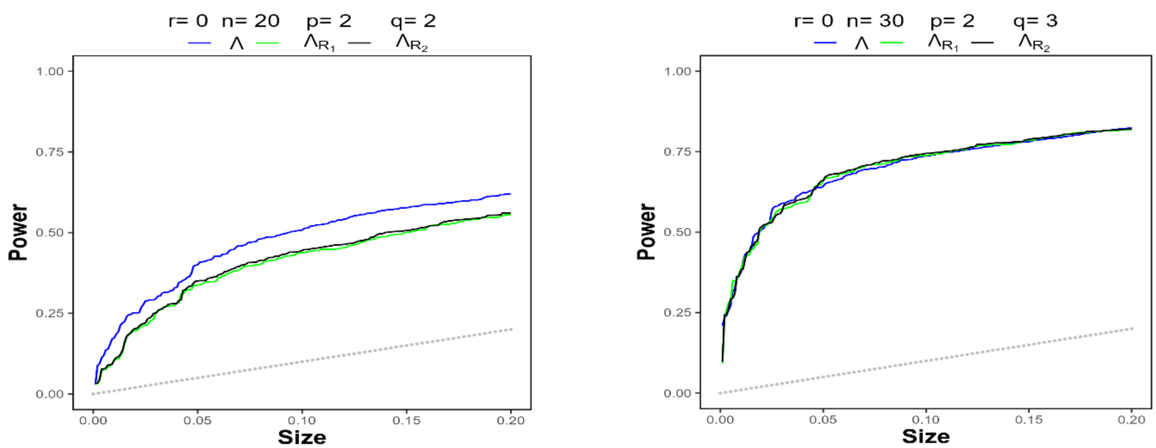


Figure 2. P-value plots of Λ, Λ_{R_1} *and* Λ_{R_2}



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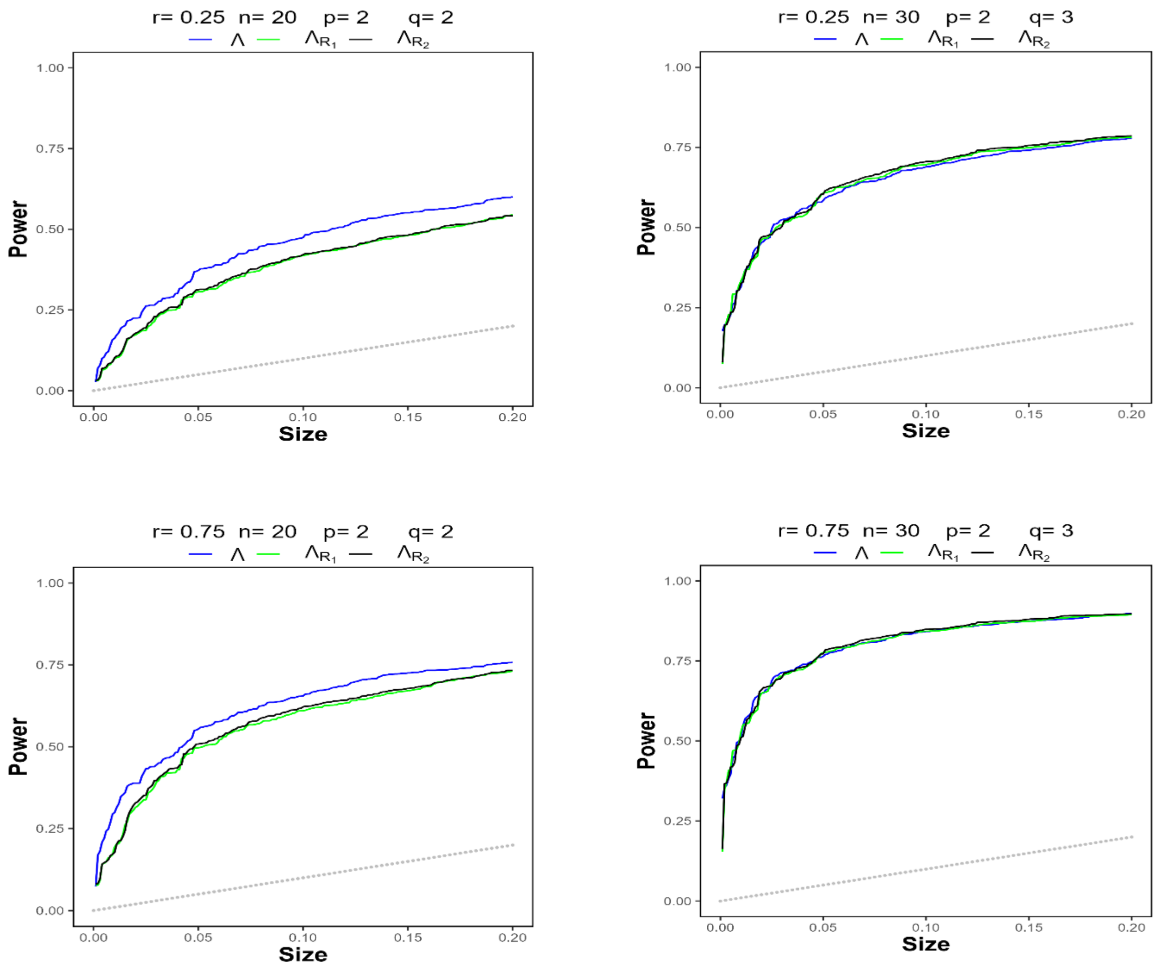
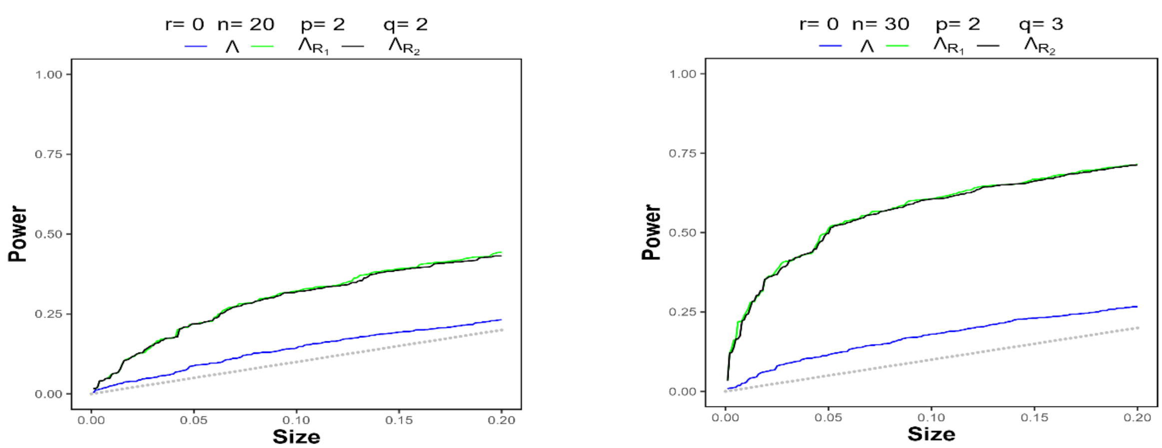


Figure 3. Curves of size power, in case of the data is fully normal



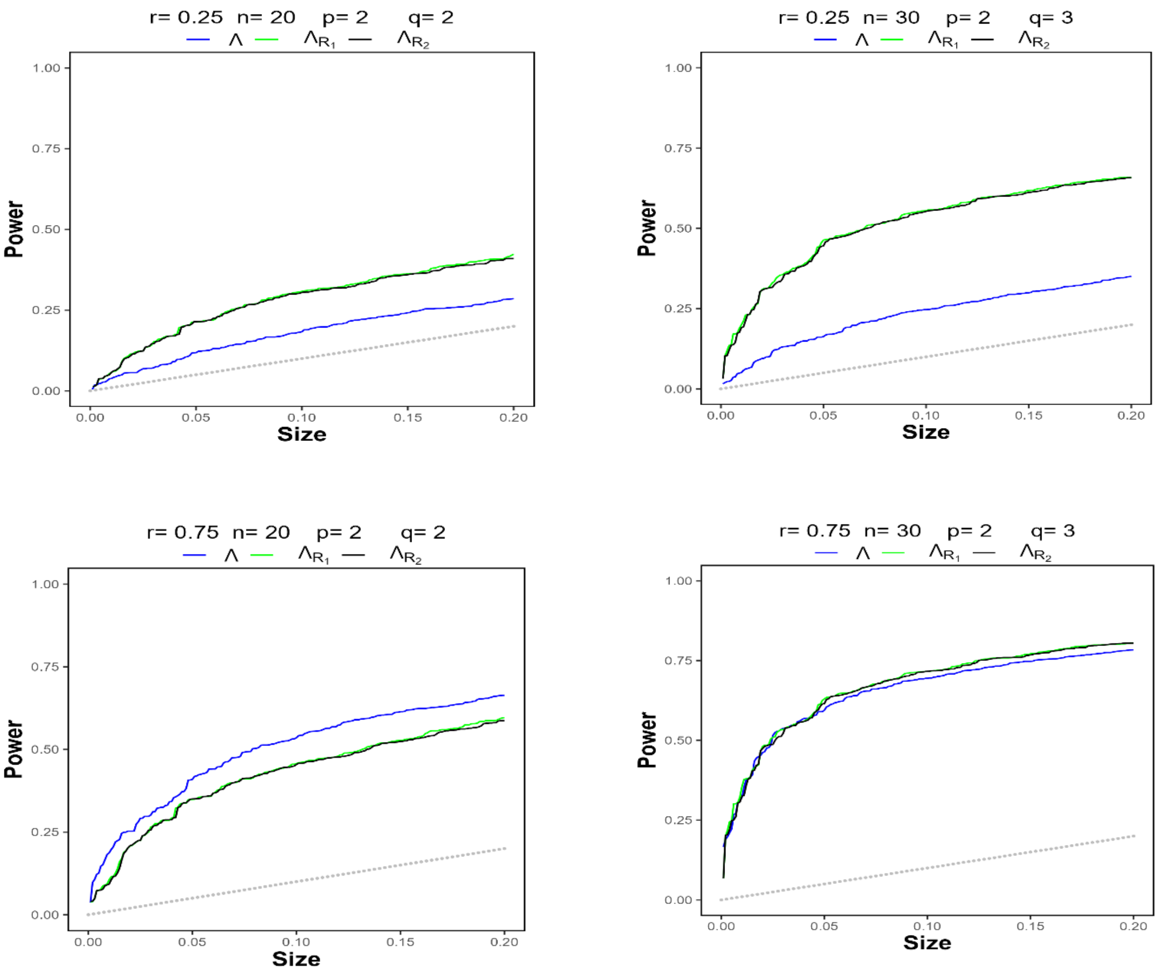
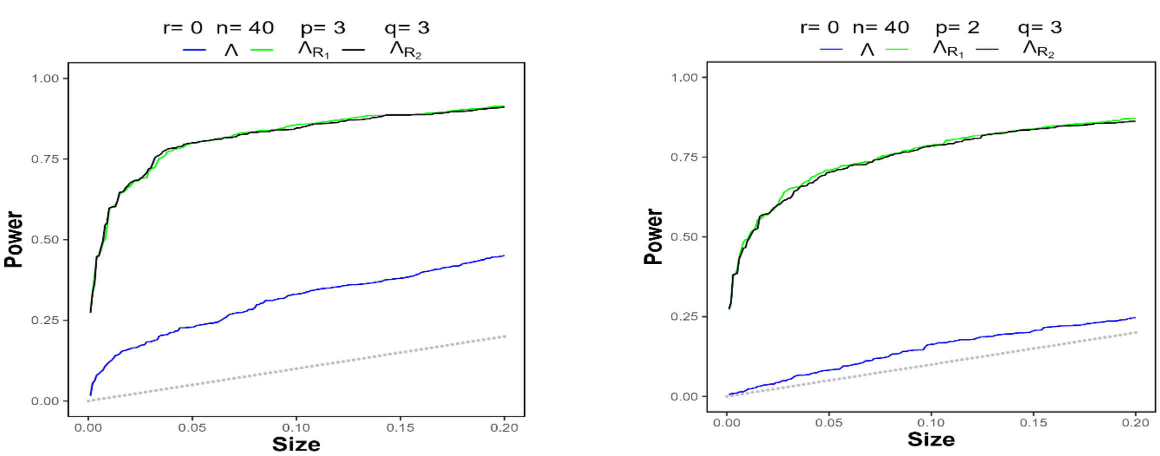


Figure 4. Curves of size power, in case of the data contains outliers



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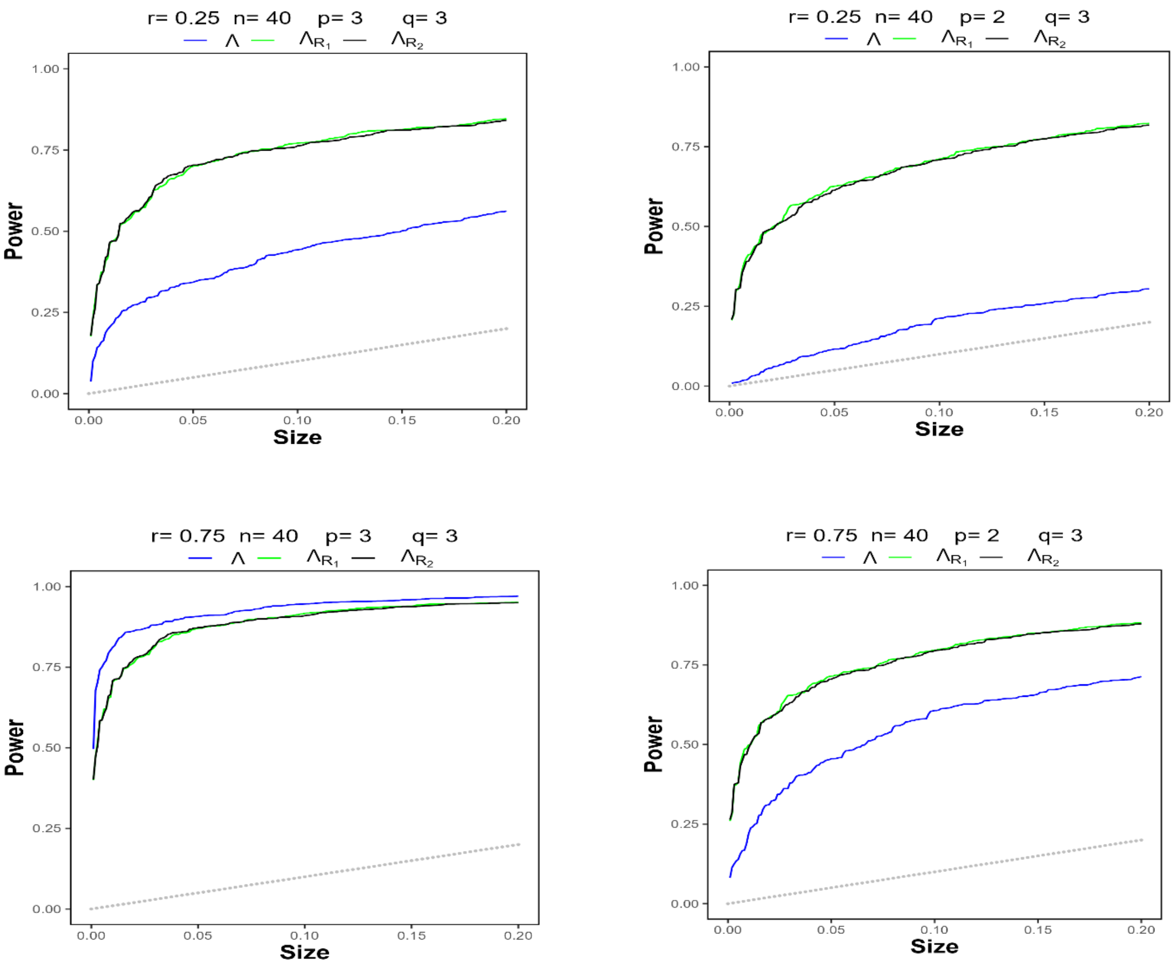

Figure 5. Curves of size power, in case of the data contains outliers

5. Conclusion:

We have presented improved robust versions of Wilks' statistic, denoted as, Λ , Λ_{R_1}, built upon the MM-estimator. Their approximate distributions were derived. The findings suggest that, under normal data distribution, the proposed statistics closely resemble the classical Wilks' statistic Λ. However, in the existence of a distribution with contaminants, the proposed statistics outperform the classical Wilks' statistic.

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مقارنه إحصائيات ويلكس الحصينة استنادًا إلى مقدر MM للانحدار الخطي المتعدد متعدد المتغيرات

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المستخلص:
في الانحدار الخطي متعدد المتغيرات، تبرز إحصائية ويلكس الكلاسيكية كطريقة مستخدمة على نطاق واسع لاختبار الفرضيات، ومع ذلك فهي تظهر حساسية عالية لتأثير القيم المتطرفة. لقد درس العديد من المؤلفين إحصائيات اختبار غير حصينه ترتكز على نظريات عادية عبر سيناريوهات متعددة. في هذا الدراسة، قمنا بتطوير طريقه حصينه لإحصائيات ويلكس، باستخدام مقدر (MM) ويعتمد هذا النهج على أوزان المشاهدات التي يتم تحديدها من خلال داله وزن هامبل وهيوبر. لقد أجرينا تحليلاً مقارناً بين الإحصائيات المقترحة وإحصائيات ويلكس الكلاسيكية. تم استخدام طريقه مونت كارلو لتقييم أداء إحصاءات الاختبار عبر مجموعات البيانات المختلفة. في حال ان البيانات تتبع التوزيع الطبيعي أظهر الطرق المقترحة معدلات خطأ من النوع الأول قريبة من مستويات الأهمية المتوقعة وقوة اختبار مشابهه للطريقة الكلاسيكية. ومع ذلك، في السيناريوهات التي تنطوي على تلوث البيانات، أظهرت الطريقة الإحصائية المقترحة أداء متفوقا. حيث ان الاحصائيات الحصينة المقترحة هي النهج المفضل عند التعامل مع البيانات المتأثرة بالقيم المتطرفة.