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Curvature Inheritance Symmetry of C_9 −manifolds

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Abstract:

This paper focused on Riemannian curvature tensor R of C_9 −manifolds. The components of covariant derivative of R determined on the space of G −structure. There are fifteen non-zero of such components and the others components given by the symmetry property and Bianchi identity of R. According to these components, the conditions on curvature tensor R of C_9 −manifolds to be has inheritance symmetry established. These conditions summarized by five equations that have common arbitrary scalar function Ψ.

Keywords: Symmetry of Riemannian spaces, Almost contact manifold, Riemannian Curvature tensor.

1. Introduction:

In 1990, Chinea and Gonzalaz classified the almost contact metric manifolds into many classes (Chinea and Gonzalaz, 1990). One of these classes is a C_9 −manifold where its geometry studied by (Rustanov et al., 2019). There are another important classes for instance, manifold of Kenmotsu type and C_{12} −manifold that introduced and examined respectively by (Abood and Abass, 2021), (Abass and Abood, 2019), (Abass and Abood, 2022) and (Abass and Al-Zamil, 2022). Moreover, a new class found by (Yusuf and Abass, 2023) that it is locally conformal of C_{12} −manifold and this class is different from locally conformal almost cosymplectic which studied recently by (Al-Hussaini, et al., 2020).

On the other side, Curvature inheritance symmetry (CI) in Riemannian spaces is defined by (Duggal, 1992). Moreover, (Salman et al., 2022) studied CI in Ricci flat spacetime, whereas (Shaikh et al., 2023) studied CI on M-projectively flat spacetimes.

This article divided into four sections. After the introduction is section 2 that devoted to reviewed the basic related definitions and theorems. In section 3, the exterior differentiation of second group of structure equations done and the components of covariant derivative of R determined on C_9 −manifold to use them in section 4. Section 4 investigated curvature inheritance symmetry on C_9 −manifold.

2. Preliminaries:

Let $\boldsymbol{M}$ be the $(\boldsymbol{2n+1})$ −dimensional manifold with $\boldsymbol{n} \in \mathbb{Z}^+$, $\boldsymbol{\nabla}$ is Levi-Civita connection, and $\boldsymbol{X(M)}$ be the $\boldsymbol{C^\infty(M)}$ −module of smooth vector fields on $\boldsymbol{M}$.

Definition 2.1 (Chinea and Gonzalez, 1990) A quadruple (η, ξ, Φ, g) of tensor fields on M is called an almost contact metric $(AC-)$ structure on M, if η is a differential 1 −form, ξ is a vector field named the characteristic vector field, Φ is a $(1,1)$ −tensor field named the structure endomorphism of the module $X(M)$, and $g = \langle\cdot,\cdot\rangle$ is a Riemannian metric, such that the following satisfied:

$$i)\ \eta(\xi) = 1;\ \ ii)\ \eta \circ \Phi = 0;\ \ iii)\ \Phi(\xi) = 0;\ \ iv)\ \Phi^2 = -id + \eta \otimes \xi;$$

$$v)\ \langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),\ \forall\ X, Y \in X(M).$$

Additionally, a manifold M equipped with an AC −structure (η, ξ, Φ, g) is called an AC −mani -fold.

Definition 2.2 (Rustanov et al., 2019) An AC −manifold M that satisfies the following identity:

$$\nabla_X(\Phi)Y = \eta(Y)\nabla_{\Phi X}\xi - \langle \Phi X, \nabla_Y \xi\rangle\xi, \qquad for\ all\ X, Y \in X(M),$$

is called a C_9 −manifold.

Lemma 2.3 (Lee, 2013) If M is a smooth manifold and $\Lambda(M)$ is the Grassmann algebra, then there exists a unique operator $\boldsymbol{d}: \Lambda(M) \to \Lambda(M)$ called an exterior differentiation, such that the following properties hold:

1. $\boldsymbol{d}$ is linear on $\mathbb{R}$;
2. $\boldsymbol{d}(\Lambda_\alpha(M)) \subseteq \Lambda_{\alpha+1}(M)$, where $\Lambda_\alpha(M)$ is the set of all α −forms on M, $\alpha = 0,1,\cdots;$
3. $\boldsymbol{d}(\omega_1 \wedge \omega_2) = \boldsymbol{d}\,\omega_1 \wedge \omega_2 + (-1)^\alpha \omega_1 \wedge \boldsymbol{d}\,\omega_2$, where $\omega_1 \in \Lambda_\alpha(M),\ \omega_2 \in \Lambda_\beta(M)$;
4. $\boldsymbol{d}^2 = \boldsymbol{d} \circ \boldsymbol{d} = 0$;
5. If $f \in C^\infty(M)$ then $\boldsymbol{d}f(X) = X(f), \forall\ X \in X(M)$.

Notation: The range of indexes $i, j, k, l, t = 0,1,2,\cdots,2n;\ a, b, c, d, h, f = 1,2,\cdots, n;$

$\hat{\imath} = \begin{cases} i+n; & 1 \le i \le n \\ i-n; & n+1 \le i \le 2n \end{cases};\ \hat{\hat{\imath}} = i;\ \hat{0} = 0;\ \ T^{[ab]} = \frac{1}{2}(T^{ab} - T^{ba});\ \ T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba});\ T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}).$

Proposition 2.4 (Rustanov et al., 2019) The first group of structure equations of C_9 −manifolds given by:

$$\boldsymbol{d}\omega = 0;\ d\omega^a = -\theta^a_b \wedge \omega^b + F^{ab}\,\omega_b \wedge \omega;\ \boldsymbol{d}\omega_a = \theta^b_a \wedge \omega_b + F_{ab}\,\omega^b \wedge \omega,$$

where: $F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}};\ \ F_{ab} = -\sqrt{-1}\Phi^0_{a,b};\ \ F^{ab} = F^{ba};\ \ F_{ab} = F_{ba};\ \ F^{ab} = \overline{F_{ab}}$. Whereas, $\{\omega^i\}$ and $\{\theta^j_i\}$ are components of the displacement forms and Riemannian connection ∇, respectively.

Theorem 2.5 (Rustanov et al., 2019) The second group of structure equations of C_9 −manifolds given by:

1. $\boldsymbol{d}\theta^a_b = -\theta^a_c \wedge \theta^c_b + A^{ad}_{bc}\,\omega^c \wedge \omega_d - F_{bc}{}^a\,\omega^c \wedge \omega + F^{ac}{}_b\,\omega_c \wedge \omega;$
2. $\boldsymbol{d}F^{ab} = -F^{cb}\,\theta^a_c - F^{ac}\,\theta^b_c + F^{abc}\,\omega_c + F^{ab}_c\,\omega^c + F^{ab0}\,\omega;$
3. $\boldsymbol{d}F_{ab} = F_{cb}\,\theta^c_a + F_{ac}\,\theta^c_b + F_{abc}\,\omega^c + F_{ab}{}^c\,\omega_c + F^{ab0}\,\omega,$

where $A^{[ad]}_{bc} = A^{ad}_{[bc]} = 0$ and $F^{a[bc]} = F_{a[bc]} = 0$

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Theorem 2.6 (Rustanov et al., 2019) The components of connection forms on associated G −structure (AG −structure) space of C_9 −manifolds are given by:

$$\theta_0^a = -F^{ab}\,\omega_b;\ \theta_a^0 = F_{ab}\,\omega^b;\ \theta_{\hat{b}}^a = 0;\ \ \theta_j^i + \theta_{\hat{\imath}}^{\hat{\jmath}} = 0.$$

Theorem 2.7 (Rustanov et al., 2019) The components of Riemann-Christoffel tensor R of type (3,1) of C_9 −manifild are determined as follow:

$$R_{a\hat{b}0}^0 = F_{ac}F^{cb};\ R_{ab0}^0 = -F_{ab0};\ R_{ab\hat{c}}^0 = -F_{ab}{}^c;\ R_{bc\hat{d}}^a = A_{bc}^{ad} + F^{ad}F_{bc};\ R_{bcd}^{\hat{a}} = -2F_{a[c}F_{|b|d]},$$

and the other components are identical zero, or deduced by the following properties:

$$1.\ R_{ijkl} = R_{jkl}^{\hat{\imath}};\ 2.\ -R_{ijlk} = R_{ijkl} = -R_{jikl};\ 3.\ R_{ijkl} = R_{klij};\ 4.\ \overline{R_{ijkl}} = R_{\hat{\imath}\hat{\jmath}\hat{k}\hat{l}};$$

$$5.\ R_{ijkl} + R_{iklj} + R_{iljk} = 0 = R_{ijkl} + R_{kijl} + R_{jkil}.$$

Proposition 2.8 (Rustanov et al., 2019) On AG −structure space for any AC −manifold, ξ^i has the the following values: $\xi^0 = 1, \xi^a = 0,$ *and* $\xi^{\hat{a}} = 0$.

Proposition 2.9 (Rustanov et al., 2019) On C_9 −manifold M, $\xi_{,j}^i$ has the the following values:

$$\xi_{,0}^0 = \xi_{,0}^a = \xi_{,a}^0 = \xi_{,b}^a = 0,\ \xi_{,\hat{b}}^a = -F^{ab},\ and\ \overline{\xi_{,j}^i} = \xi_{,\hat{\jmath}}^{\hat{\imath}}.$$

Definition 2.10 (Lee, 2013) Suppose M is a smooth manifold, V is a smooth vector field on M, and Θ is the flow of V. For any smooth vector field W on M, define a rough vector field on M, denoted by $\mathcal{L}_V W$ and called the Lie derivative of W with respect to V, by

$$(\mathcal{L}_V W)_p = \frac{d}{d\tau}|_{\tau=0}\, d(\Theta_{-\tau})_{\Theta_\tau(p)}(W_{\Theta_\tau(p)}) = \lim_{\tau\to\infty} \frac{d(\Theta_{-\tau})_{\Theta_\tau(p)}(W_{\Theta_\tau(p)}) - W_p}{\tau}, \quad (1)$$

provided the derivative exists. For small $\tau \neq 0$, at least the difference quotient makes sense: Θ_τ is defined in a neighbourhood of $p \in M$, and $\Theta_{-\tau}$ is the inverse of Θ_τ, so both $d(\Theta_{-\tau})_{\Theta_\tau(p)}(W_{\Theta_\tau(p)})$ and W_p are elements of the tangent space $\mathcal{T}_p(M)$.

Lemma 2.11 (Kirichenko and Kharitonova, 2012**)** If R is Riemannian curvature tensor of type (4,0), then the components of its covariant derivative on space of AG −structure satisfy the relation:

$$\boldsymbol{d}R_{ijkl} - R_{tjkl}\ \theta_i^t - R_{itkl}\ \theta_j^t - R_{ijtl}\ \theta_k^t - R_{ijkt}\ \theta_l^t = R_{ijkl,t}\ \omega^t \quad (2)$$

Definition 2.12 (Salman,et al. 2022) The curvature tensor R on Riemannian manifold (M, g) is called inheritance along a vector field ξ, if R satisfies the following:

$$\mathcal{L}_\xi R = 2\Psi R, \quad (3)$$

where Ψ is a scalar function. Moreover, the equation (3) can be written in local coordinates as:

$$R_{jkl,t}^i\,\xi^t - R_{jkl}^t\,\xi_{,t}^i + R_{tkl}^i\,\xi_{,j}^t + R_{jtl}^i\,\xi_{,k}^t + R_{jkt}^i\,\xi_{,l}^t = 2\Psi\,R_{jkl}^i \quad (4)$$

3. Covariant Derivative of Curvature Tensor:

In this section, the exterior differentiation of second group of structure equations done and the components of covariant derivative of R determined on C_9 −manifold.

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Theorem 3.1 On AG −structure of C_9 −manifold, There exist smooth functions, such that the following equalities absolutely verified:

1. $\boldsymbol{d}A^{ad}_{bc} = A^{ad}_{hc}\ \theta^h_b + A^{ad}_{bh}\ \theta^h_c - A^{hd}_{bc}\ \theta^a_h - A^{ah}_{bc}\ \theta^d_h + A^{ad}_{bch}\ \omega^h + A^{adh}_{bc}\ \omega_h + A^{ad}_{bc0}\ \omega;$
2. $\boldsymbol{d}F_{ab}{}^{c} = F_{hb}{}^{c}\ \theta^h_a + F_{ah}{}^{c}\ \theta^h_b - F_{ab}{}^{h}\ \theta^c_h + F_{ab}{}^{c}{}_{h}\ \omega^h + F_{ab}{}^{ch}\ \omega_h + F_{ab}{}^{c0}\ \omega;$
3. $\boldsymbol{d}F^{ab}{}_{c} = F^{ab}{}_{h}\ \theta^h_c - F^{ah}{}_{c}\ \theta^b_h - F^{hb}{}_{c}\ \theta^a_h + F^{ab}{}_{ch}\ \omega^h + F^{ab}{}_{c}{}^{h}\ \omega_h + F^{ab}{}_{c0}\ \omega;$
4. $\boldsymbol{d}F^{abc} = -F^{abh}\ \theta^c_h - F^{ahc}\ \theta^b_h - F^{hbc}\ \theta^a_h + F^{abch}\ \omega_h + F^{abc}{}_{h}\ \omega^h + F^{abc0}\ \omega;$
5. $\boldsymbol{d}F_{abc} = F_{abh}\ \theta^h_c + F_{ahc}\ \theta^h_b + F_{hbc}\ \theta^h_a + F_{abch}\ \omega^h + F_{abc}{}^{h}\ \omega_h + F_{abc0}\ \omega;$
6. $\boldsymbol{d}F^{ab0} = -F^{ah0}\ \theta^b_h - F^{hb0}\ \theta^a_h + F^{ab0h}\ \omega_h + F^{ab0}{}_{h}\ \omega^h + F^{ab00}\ \omega;$
7. $\boldsymbol{d}F_{ab0} = F_{ah0}\ \theta^h_b + F_{hb0}\ \theta^h_a + F_{ab0h}\ \omega^h + F_{ab0}{}^{h}\ \omega_h + F_{ab00}\ \omega,$

where $A^{a[dh]}_{bc} = A^{ad}_{b[ch]} = F^{ab[ch]} = F^{ab}{}_{[ch]} = 0;\ A^{ad}_{bc0} + F^{ad}{}_{bc} + F_{bc}{}^{ad} = 0;$

$$F^{a[c}{}_{b}{}^{h]} - A^{a[c}_{bd}\, F^{h]d} = 0;\quad A^{ad}_{b[c}\, F_{h]d} - F_{b[c}{}^{a}{}_{h]} = 0;\quad F^{abc}{}_{h} - F^{ab}{}_{h}{}^{c} - F^{ad}\, A^{bc}_{dh} - F^{bd}\, A^{ac}_{dh} = 0;$$

$$F^{ab0}{}_{c} + F^{abh}\, F_{hc} - F^{ab}{}_{c}{}_{0} + F^{hb}\, F_{hc}{}^{a} + F^{ah}\, F_{hc}{}^{b} = 0;$$

$$F^{ab0c} + F^{ab}{}_{h}\, F^{hc} - F^{abc0} - F^{hb}\, F^{ac}{}_{h} - F^{ah}\, F^{bc}{}_{h} = 0.$$

Proof. By taken operator $\boldsymbol{d}$ for theorem 2.5; item 1, we get on $\boldsymbol{d}A^{ad}_{bc},\ \boldsymbol{d}F_{ab}{}^{c}$ *and* $\boldsymbol{d}F^{ab}{}_{c}$ as follow:

$$\begin{aligned}\boldsymbol{d}^2\theta^a_b + d\theta^a_c \wedge \theta^c_b - \theta^a_c \wedge d\theta^c_b \\ = \boldsymbol{d}A^{ad}_{bc} \wedge \omega^c \wedge \omega_d + A^{ad}_{bc}\, \boldsymbol{d}\omega^c \wedge \omega_d - A^{ad}_{bc}\, \omega^c \wedge \boldsymbol{d}\omega_d - \boldsymbol{d}F_{bc}{}^{a} \wedge \omega^c \wedge \omega \\ -F_{bc}{}^{a}\, \boldsymbol{d}\omega^c \wedge \omega + F_{bc}{}^{a}\, \omega^c \wedge \boldsymbol{d}\omega + \boldsymbol{d}F^{ac}{}_{b} \wedge \omega_c \wedge \omega + F^{ac}{}_{b}\, \boldsymbol{d}\omega_c \wedge \omega \\ -F^{ac}{}_{a}\, \omega_c \wedge \boldsymbol{d}\omega.\end{aligned}$$

Now, from Lemma 2.3, Proposition 2.4, and Theorem 2.5 and by reorder last equation, we obtain:

$$\begin{aligned}0 = (\boldsymbol{d}A^{ad}_{bc} + A^{hd}_{bc}\ \theta^a_h - A^{ad}_{hc}\ \theta^h_b + A^{ah}_{bc}\ \theta^d_h - A^{ad}_{bh}\ \theta^h_c) \wedge \omega^c \wedge \omega_d \\ -(\boldsymbol{d}F_{bc}{}^{a} + F_{bc}{}^{h}\, \theta^a_h - F_{hc}{}^{a}\, \theta^h_b - F_{ch}{}^{a}\, \theta^h_c) \wedge \omega^c \wedge \omega \\ +(\boldsymbol{d}F^{ac}{}_{b} + F^{hc}{}_{b}\, \theta^a_h + F^{ah}{}_{b}\, \theta^c_h - F^{ac}{}_{h}\, \theta^h_b) \wedge \omega_c \wedge \omega \\ +A^{a[d}_{bc}\, F^{h]c}\, \omega_d \wedge \omega_h \wedge \omega - A^{ad}_{b[c}F_{h]d}\, \omega^c \wedge \omega^h \wedge \omega.\end{aligned} \tag{5}$$

Since each of $\boldsymbol{d}A^{ad}_{bc} + A^{hd}_{bc}\ \theta^a_h - A^{ad}_{hc}\ \theta^h_b + A^{ah}_{bc}\ \theta^d_h - A^{ad}_{bh}\ \theta^h_c,\ \boldsymbol{d}F_{bc}{}^{a} + F_{bc}{}^{h}\, \theta^a_h - F_{hc}{}^{a}\, \theta^h_b - F_{ch}{}^{a}\, \theta^h_c$, and $\boldsymbol{d}F^{ac}{}_{b} + F^{hc}{}_{b}\, \theta^a_h + F^{ah}{}_{b}\, \theta^c_h - F^{ac}{}_{h}\, \theta^h_b$ is 1 −form, then them can be written according to the family of basis for 1 −forms on AG −structure space $\{\theta^h_f, \omega^h, \omega_h, \omega\}$ as follow:

$$\boldsymbol{d}A^{ad}_{bc} + A^{hd}_{bc}\ \theta^a_h - A^{ad}_{hc}\ \theta^h_b + A^{ah}_{bc}\ \theta^d_h - A^{ad}_{bh}\, \theta^h_c = A^{adf}_{bch}\, \theta^h_f + A^{ad}_{bch}\, \omega^h + A^{adh}_{bc}\, \omega_h + A^{ad}_{bc0}\, \omega, \tag{6}$$

$$\boldsymbol{d}F_{bc}{}^{a} + F_{bc}{}^{h}\, \theta^a_h - F_{hc}{}^{a}\, \theta^h_b - F_{ch}{}^{a}\, \theta^h_c = F_{bc}{}^{af}{}_{h}\, \theta^h_f + F_{bc}{}^{a}{}_{h}\, \omega^h + F_{bc}{}^{ah}\, \omega_h + F_{bc}{}^{a0}\, \omega, \tag{7}$$

$$\boldsymbol{d}F^{ac}{}_{b} + F^{hc}{}_{b}\, \theta^a_h + F^{ah}{}_{b}\, \theta^c_h - F^{ac}{}_{h}\, \theta^h_b = F^{ac}{}_{bh}{}^{f}\, \theta^h_f + F^{ac}{}_{bh}\, \omega^h + F^{ac}{}_{b}{}^{h}\, \omega_h + F^{ac}{}_{b0}\, \omega, \tag{8}$$

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where $\{A^{adf}_{bch}, A^{ad}_{bch}, A^{adh}_{bc}, A^{ad}_{bc0}\}, \{F_{bc}{}^{af}{}_{h}, F_{bc}{}^{a}{}_{h}, F_{bc}{}^{ah}, F_{bc}{}^{a0}\}$ and $\{F^{ac}{}_{bh}{}^{f}, F^{ac}{}_{bh}, F^{ac}{}_{b}{}^{h}, F^{ac}{}_{b0}\}$ are appropriate families of smooth functions. Then the equation (5) be as follow:

$$0 = A^{adf}_{bch}\,\theta^h_f \wedge \omega^c \wedge \omega_d + A^{ad}_{b[ch]}\,\omega^h \wedge \omega^c \wedge \omega_d + A^{a[dh]}_{bc}\,\omega_h \wedge \omega^c \wedge \omega_d$$

$$+A^{ad}_{bc0}\,\omega \wedge \omega^c \wedge \omega_d$$

$$-F_{bc}{}^{af}{}_{h}\,\theta^h_f \wedge \omega^c \wedge \omega - F_{b[c}{}^{a}{}_{h]}\,\omega^h \wedge \omega^c \wedge \omega - F_{bc}{}^{ah}\,\omega_h \wedge \omega^c \wedge \omega$$

$$+F^{ac}{}_{bh}{}^{f}\,\theta^h_f \wedge \omega_c \wedge \omega + F^{ac}{}_{bh}\,\omega^h \wedge \omega_c \wedge \omega + F^{a[c}{}_{b}{}^{h]}\,\omega_h \wedge \omega_c \wedge \omega$$

$$+A^{a[d}_{bc}F^{h]c}\,\omega_d \wedge \omega_h \wedge \omega - A^{ad}_{b[c}F_{h]d}\,\omega^c \wedge \omega^h \wedge \omega.$$

So, the last equation gives the following relations by changes some indexes and uses the fact:
$\omega^i \wedge \omega^j = -\omega^j \wedge \omega^i$ (note that $\overline{\omega^i} = \omega^{\hat{\imath}} = \omega_i$ and $\omega^0 = \omega$)

$$A^{adf}_{bch} = F_{bc}{}^{af}{}_{h} = F^{ac}{}_{bh}{}^{f} = A^{a[dh]}_{bc} = A^{ad}_{b[ch]} = 0;$$

$$A^{ad}_{bc0} + F^{ad}{}_{bc} + F_{bc}{}^{ad} = 0; \quad F^{a[c}{}_{b}{}^{h]} - A^{a[c}_{bd}F^{h]d} = 0; \quad A^{ad}_{b[c}F_{h]d} - F_{b[c}{}^{a}{}_{h]} = 0.$$

Now, by using the above relations with equations (6), (7), and (8), we get the result of this theorem in items 1; 2; and 3.

In the same way, we can obtain the items $4-7$ by applying the exterior differentiation operator $\boldsymbol{d}$ on Theorem 2.5; items 2 and 3 and using Lemma 2.3, Proposition 2.4, Theorem 2.5, and changing the indexes of some terms to obtain:

$$0 = (dF^{abc} + F^{hbc}\theta^a_h + F^{ahc}\theta^b_h + F^{abh}\theta^c_h) \wedge \omega_c$$

$$+(dF^{ab0} + F^{cb0}\theta^a_c + F^{ac0}\theta^b_c) \wedge \omega + F^{ab}{}_{[ch]}\,\omega^h \wedge \omega^c$$

$$+(F^{ab}{}_{c}{}^{h} + F^{ad}\,A^{bh}_{dc} + F^{bd}\,A^{ah}_{dc})\,\omega_h \wedge \omega^c$$

$$+(F^{abh}F_{hc} - F^{ab}{}_{c}{}_{0} + F^{hb}F_{hc}{}^{a} + F^{ah}F_{hc}{}^{b})\,\omega^c \wedge \omega$$

$$+(F^{ab}{}_{h}F^{hc} - F^{hb}F^{ac}{}_{h} - F^{ah}F^{bc}{}_{h})\,\omega_c \wedge \omega, \tag{9}$$

and

$$0 = (dF_{abc} - F_{hbc}\theta^h_a - F_{ahc}\theta^h_b - F_{abh}\theta^h_c) \wedge \omega^c$$

$$+(dF_{ab0} - F_{cb0}\theta^c_a - F_{ac0}\theta^c_b) \wedge \omega + F_{ab}{}^{[ch]}\,\omega_h \wedge \omega_c$$

$$+(F_{ab}{}^{c}{}_{h} + F_{ad}\,A^{dc}_{bh} + F_{bd}\,A^{dc}_{ah})\,\omega^h \wedge \omega_c$$

$$+(F_{abh}F^{hc} - F_{ab}{}^{c}{}^{0} + F_{hb}F^{hc}_{a} + F_{ah}F^{hc}_{b})\,\omega_c \wedge \omega$$

$$+(F_{ab}{}^{h}F_{hc} - F_{hb}F_{ac}{}^{h} - F_{ah}F_{bc}{}^{h})\,\omega^c \wedge \omega. \tag{10}$$

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Since each of $\boldsymbol{d}F^{abc} + F^{hbc}\,\theta_h^a + F^{ahc}\,\theta_h^b + F^{abh}\,\theta_h^c$, $\boldsymbol{d}F_{abc} - F_{hbc}\,\theta_a^h - F_{ahc}\,\theta_b^h - F_{abh}\,\theta_c^h$,

$\boldsymbol{d}F^{ab0} + F^{cb0}\theta_c^a + F^{ac0}\theta_c^b$, and $\boldsymbol{d}F_{ab0} - F_{cb0}\theta_a^c - F_{ac0}\theta_b^c$ is $1-$form, then we can write them according to the family of basis for $1-$forms on $AG-$structure space $\{\theta_f^h, \omega^h, \omega_h, \omega\}$ as follow:

$$\boldsymbol{d}F^{abc} + F^{hbc}\theta_h^a + F^{ahc}\theta_h^b + F^{abh}\theta_h^c = F^{abcf}{}_h\,\theta_f^h + F^{abc}{}_h\,\omega^h + F^{abch}\omega_h + F^{abc0}\omega, \quad (11)$$

$$\boldsymbol{d}F_{abc} - F_{hbc}\theta_a^h - F_{ahc}\theta_b^h - F_{abh}\theta_c^h = F_{abch}^{\ \ f}\,\theta_f^h + F_{abch}\,\omega^h + F_{abc}{}^h\,\omega_h + F_{abc0}\,\omega, \quad (12)$$

$$\boldsymbol{d}F^{ab0} + F^{cb0}\theta_c^a + F^{ac0}\theta_c^b = F^{ab0f}{}_h\,\theta_f^h + F^{ab0}{}_h\,\omega^h + F^{ab0h}\,\omega_h + F^{ab00}\,\omega, \quad (13)$$

$$\boldsymbol{d}F_{ab0} - F_{cb0}\theta_a^c - F_{ac0}\theta_b^c = F_{ab0h}^{\ \ f}\,\theta_f^h + F_{ab0h}\,\omega^h + F_{ab0}{}^h\,\omega_h + F_{ab00}\,\omega. \quad (14)$$

Where $\{F^{abcf}{}_h, F^{abc}{}_h, F^{abch}, F^{abc0}\}$, $\{F_{abch}^{\ \ f}, F_{abc}{}^h, F_{abch}, F_{abc0}\}$, $\{F^{ab0f}{}_h, F^{ab0}{}_h, F^{ab0h}, F^{ab00}\}$, and $\{F_{ab0h}^{\ \ f}, F_{ab0}{}^h, F_{ab0h}, F_{ab00}\}$ are appropriate families of smooth functions. Then the equations (9) and (10) be as follow:

$$\begin{aligned}
0 &= (F^{abcf}{}_h\,\theta_f^h + F^{abc}{}_h\,\omega^h + F^{abch}\omega_h + F^{abc0}\omega) \wedge \omega_c \\
&+(F^{ab0f}{}_h\,\theta_f^h + F^{ab0}{}_h\,\omega^h + F^{ab0h}\,\omega_h) \wedge \omega + F^{ab}{}_{[ch]}\,\omega^h \wedge \omega^c \\
&+(F^{ab}{}_c{}^h + F^{ad}\,A_{dc}^{bh} + F^{bd}\,A_{dc}^{ah})\,\omega_h \wedge \omega^c \\
&+(F^{abh}F_{hc} - F^{ab}{}_{c0} + F^{hb}F_{hc}{}^a + F^{ah}F_{hc}{}^b)\,\omega^c \wedge \omega \\
&+(F^{ab}{}_h F^{hc} - F^{hb}F^{ac}{}_h - F^{ah}F^{bc}{}_h)\,\omega_c \wedge \omega \\
&= (F^{abcf}{}_h\,\theta_f^h \wedge \omega_c + F^{abc}{}_h\,\omega^h \wedge \omega_c + F^{ab[ch]}\omega_h \wedge \omega_c + F^{abc0}\omega \wedge \omega_c) \\
&+(F^{ab0f}{}_h\,\theta_f^h \wedge \omega + F^{ab0}{}_h\,\omega^h \wedge \omega + F^{ab0h}\,\omega_h \wedge \omega) + F^{ab}{}_{[ch]}\,\omega^h \wedge \omega^c \\
&+(F^{ab}{}_c{}^h\,\omega_h \wedge \omega^c + F^{ad}\,A_{dc}^{bh}\,\omega_h \wedge \omega^c + F^{bd}\,A_{dc}^{ah}\,\omega_h \wedge \omega^c) \\
&+(F^{abh}F_{hc}\,\omega^c \wedge \omega - F^{ab}{}_{c\ 0}\,\omega^c \wedge \omega + F^{hb}F_{hc}{}^a\,\omega^c \wedge \omega + F^{ah}F_{hc}{}^b\,\omega^c \wedge \omega) \\
&+(F^{ab}{}_h F^{hc} - F^{hb}F^{ac}{}_h - F^{ah}F^{bc}{}_h)\,\omega_c \wedge \omega,
\end{aligned}$$

and

$$\begin{aligned}
0 &= (F_{abch}^{\ \ f}\,\theta_f^h + F_{abch}\,\omega^h + F_{abc}{}^h\,\omega_h + F_{abc0}\,\omega) \wedge \omega^c \\
&+(F_{ab0h}^{\ \ f}\,\theta_f^h + F_{ab0h}\,\omega^h + F_{ab0}{}^h\,\omega_h) \wedge \omega + F_{ab}{}^{[ch]}\,\omega_h \wedge \omega_c \\
&+(F_{ab}{}^c{}_h + F_{ad}\,A_{bh}^{dc} + F_{bd}\,A_{ah}^{dc})\,\omega^h \wedge \omega_c \\
&+(F_{abh}F^{hc} - F_{ab}{}^{c\ 0} + F_{hb}F^{hc}{}_a + F_{ah}F^{hc}{}_b)\,\omega_c \wedge \omega
\end{aligned}$$

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$$+(F_{ab}{}^{h}F_{hc} - F_{hb}F_{ac}{}^{h} - F_{ah}F_{bc}{}^{h})\,\omega^c \wedge \omega$$

$$= (F_{abch}{}^{f}\,\theta_f^h \wedge \omega^c + F_{ab[ch]}\,\omega^h \wedge \omega^c + F_{abc}{}^{h}\,\omega_h \wedge \omega^c + F_{abc0}\,\omega \wedge \omega^c)$$

$$+(F_{ab0h}{}^{f}\,\theta_f^h \wedge \omega + F_{ab0h}\,\omega^h \wedge \omega + F_{ab0}{}^{h}\,\omega_h \wedge \omega) + F_{ab}{}^{[ch]}\,\omega_h \wedge \omega_c$$

$$+(F_{ab}{}^{c}{}_{h}\,\omega^h \wedge \omega_c + F_{ad}\,A_{bh}^{dc}\,\omega^h \wedge \omega_c + F_{bd}\,A_{ah}^{dc}\,\omega^h \wedge \omega_c)$$

$$+(F_{abh}F^{hc}\,\omega_c \wedge \omega - F_{ab}{}^{c0}\,\omega_c \wedge \omega + F_{hb}F^{hc}{}_{a}\,\omega_c \wedge \omega + F_{ah}F^{hc}{}_{b}\,\omega_c \wedge \omega)$$

$$+(F_{ab}{}^{h}F_{hc}\,\omega^c \wedge \omega - F_{hb}F_{ac}{}^{h}\,\omega^c \wedge \omega - F_{ah}F_{bc}{}^{h}\,\omega^c \wedge \omega).$$

These equations given us the next relations after changing indexes of some terms:

$$0 = F_{abch}{}^{f} = F^{abcf}{}_{h}; \qquad 0 = F_{ab0h}{}^{f} = F^{ab0f}{}_{h};$$

$$0 = F_{ab[ch]} = F^{ab[ch]}; \quad 0 = F_{ab}{}^{[ch]} = F^{ab}{}_{[ch]};$$

$$0 = F_{abc}{}^{h} - F_{ab}{}^{h}{}_{c} - F_{ad}\,A_{bc}^{dh} - F_{bd}\,A_{ac}^{dh};$$

$$0 = F^{abc}{}_{h} - F^{ab}{}_{h}{}^{c} - F^{ad}\,A_{dh}^{bc} - F^{bd}\,A_{dh}^{ac};$$

$$0 = F_{abc0} - F_{ab0c} - F_{ab}{}^{h}F_{hc} + F_{hb}F_{ac}{}^{h} + F_{ah}F_{bc}{}^{h};$$

$$0 = F^{abc0} - F^{ab0c} - F^{ab}{}_{h}F^{hc} + F^{hb}F^{ac}{}_{h} + F^{ah}F^{bc}{}_{h};$$

$$0 = F_{ab0}{}^{c} + F_{abh}F^{hc} - F_{ab}{}^{c0} + F_{hb}F^{hc}{}_{a} + F_{ah}F^{hc}{}_{b};$$

$$0 = F^{ab0}{}_{c} + F^{abh}F_{hc} - F^{ab}{}_{c0} + F^{hb}F_{hc}{}^{a} + F^{ah}F_{hc}{}^{b}.$$

Now, by using the last relations with equations (11), (12), (13), and (14), we obtain the result. ■

Theorem 3.2 On the AG −structure space for the C_9 −manifold, the components of ∇R are given by:

1. $R_{0a\hat{b}0,0} = F_{ac0}\,F^{cb} + F_{ac}\,F^{cb0}$;
2. $R_{0a\hat{b}0,h} = F_{ach}F^{cb} + F_{ac}\,F^{cb}{}_{h} + F^{bf}{}_{a}\,F_{fh}$;
3. $R_{0a\hat{b}0,\hat{h}} = F_{ac}{}^{h}\,F^{cb} + F_{ac}\,F^{cbh} + F_{af}{}^{b}\,F^{fh}$;
4. $R_{0ab0,0} = -F_{ab00}$;
5. $R_{0ab0,h} = -(F_{ab0h} + 2F_{ba}{}^{f}\,F_{fh})$;
6. $R_{0ab0,\hat{h}} = -F_{ab0}{}^{h}$;
7. $R_{0bc\hat{d},0} = -F_{bc}{}^{d0}$;
8. $R_{0bc\hat{d},h} = F_{bf}\,F^{fd}\,F_{ch} - F_{bc}{}^{d}{}_{h} + (A_{bc}^{ad} + F^{ad}F_{bc})\,F_{ah}$;
9. $R_{0bc\hat{d},\hat{h}} = F_{bc0}\,F^{dh} - F_{bc}{}^{dh}$;
10. $R_{\hat{a}bc\hat{d},0} = A_{bc0}^{ad} + F^{ad0}F_{bc} + F^{ad}\,F_{bc0}$;
11. $R_{\hat{a}bc\hat{d},h} = A_{bch}^{ad} + F^{ad}{}_{h}\,F_{bc} + F^{ad}\,F_{bch} + F^{ad}{}_{c}\,F_{bh} + F^{da}{}_{b}\,F_{ch}$;

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12. $R_{\hat{a}bc\hat{d},\hat{h}} = A^{adh}_{bc} + F^{adh}F_{bc} + F^{ad}\,{F_{bc}}^{h} + {F_{bc}}^{d}\,F^{ah} + {F_{cb}}^{a}\,F^{dh}$;
13. $R_{abcd,0} = F_{ad0}\,F_{bc} + F_{ad}\,F_{bc0} - F_{ac0}\,F_{bd} - F_{ac}\,F_{bd0}$;
14. $R_{abcd,h} = F_{adh}\,F_{bc} + F_{ad}\,F_{bch} - F_{ach}\,F_{bd} - F_{ac}\,F_{bdh}$;
15. $R_{abcd,\hat{h}} = {F_{ad}}^{h}\,F_{bc} + F_{ad}\,{F_{bc}}^{h} - {F_{ac}}^{h}\,F_{bd} - F_{ac}\,{F_{bd}}^{h}$,

and the remaining components are identical to zero or can determined by the properties of R in Theorem 2.7.

Proof. The results obtain from equation (2) by taking:

$$(l,i,j,k,) = (0,0,a,\hat{b}), (0,0,a,b), (\hat{c},0,a,b), (\hat{d},a,b,c), (d,\hat{a},b,c);$$

$$t = 0, h, \hat{h}$$

and look at Proposition 2.4 and Theorems 2.5, 2.6, 2.7, and 3.1

1. $(l,i,j,k) = (0,0,a,\hat{b});\quad t = 0,h,\hat{h}$

$$R_{0a\hat{b}0,0}\ \omega^0 + R_{0a\hat{b}0,h}\ \omega^h + R_{0a\hat{b}0,\hat{h}}\ \omega^{\hat{h}} = \boldsymbol{d}R_{0a\hat{b}0} - R_{\hat{h}a\hat{b}0}\ \theta_0^{\hat{h}} - R_{0h\hat{b}0}\ \theta_a^h - R_{0a\hat{h}0}\ \theta_{\hat{b}}^{\hat{h}} - R_{0a\hat{b}h}\ \theta_0^h$$

$$R_{0a\hat{b}0,0}\ \omega + R_{0a\hat{b}0,h}\ \omega^h + R_{0a\hat{b}0,\hat{h}}\ \omega_h = (\boldsymbol{d}F_{ac})\ F^{cb} + F_{ac}\ \boldsymbol{d}F^{cb}F^{bh}_{\ \ a}\ F_{hf}\omega^f - F_{hc}F^{cb}\ \theta_a^h$$
$$+F_{ac}\ F^{ch}\ \theta_h^b + {F_{ah}}^{b}\ F^{hf}\omega_f$$

$$R_{0a\hat{b}0,0}\ \omega + R_{0a\hat{b}0,h}\ \omega^h + R_{0a\hat{b}0,\hat{h}}\ \omega_h = (F_{ac0}\ F^{cb} + F_{ac}\ F^{cb0}\ )\omega + (F_{ach}F^{cb} + F_{ac}\ F^{cb}_{\ \ h}$$
$$+F^{bf}_{\ \ a}\ F_{fh})\omega^h + ({F_{ac}}^{h}\ F^{cb}$$
$$+F_{ac}\ F^{cbh} + {F_{af}}^{b}\ F^{fh})\omega_h.$$

$$R_{0a\hat{b}0,0} = F_{ac0}\ F^{cb} + F_{ac}\ F^{cb0},$$

$$R_{0a\hat{b}0,h} = F_{ach}F^{cb} + F_{ac}\ F^{cb}_{\ \ h} + F^{bf}_{\ \ a}\ F_{fh},$$

$$R_{0a\hat{b}0,\hat{h}} = {F_{ac}}^{h}\ F^{cb} + F_{ac}\ F^{cbh} + {F_{af}}^{b}\ F^{fh}.$$

2. $(l,i,j,k,) = (0,0,a,b);\quad t = 0,h,\hat{h}$

$$R_{0ab0,0}\ \omega + R_{0ab0,h}\ \omega^h + R_{0ab0,\hat{h}}\ \omega_h = \boldsymbol{d}R_{0ab0} - R_{\hat{h}ab0}\ \theta_0^{\hat{h}} - R_{0hb0}\ \theta_a^h - R_{0ah0}\ \theta_b^h - R_{0ab\hat{h}}\ \theta_0^{\hat{h}}$$
$$= -\boldsymbol{d}F_{ab0} - {F_{ba}}^{h}\ F_{hf}\omega^f + F_{hb0}\ \theta_a^h + F_{ah0}\ \theta_b^h$$
$$-{F_{ab}}^{h}F_{hf}\omega^f$$
$$= -F_{ab00}\ \omega - (F_{ab0h} + 2{F_{ba}}^{f}\ F_{fh})\omega^h - {F_{ab0}}^{h}\ \omega_h$$

$$R_{0ab0,0} = -F_{ab00},$$

$$R_{0ab0,h} = -(F_{ab0h} + 2{F_{ba}}^{f}\ F_{fh}),$$

$$R_{0ab0,\hat{h}} = -{F_{ab0}}^{h}.$$

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3. $(l,i,j,k)=(\hat{c},0,a,b);\quad t=0,h,\hat{h}$

$$R_{0ab\hat{c},0}\ \omega+R_{0ab\hat{c},h}\ \omega^{h}+R_{0ab\hat{c},\hat{h}}\ \omega_{h}=\boldsymbol{d}R_{0ab\hat{c}}-R_{\hat{h}ab\hat{c}}\ \theta_{0}^{\hat{h}}-R_{0hb\hat{c}}\ \theta_{a}^{h}-R_{0a0\hat{c}}\ \theta_{b}^{0}-R_{0ah\hat{c}}\ \theta_{b}^{h}$$

$$-R_{0ab0}\ \theta_{\hat{c}}^{0}-R_{0ab\hat{h}}\ \theta_{\hat{c}}^{\hat{h}}$$

$$=-\boldsymbol{d}F_{ab}{}^{c}+(A_{ab}^{hc}+F^{hc}F_{ab})\ F_{hf}\omega^{f}+F_{hb}{}^{c}\ \theta_{a}^{h}$$

$$+F_{af}F^{fc}\ F_{bh}\omega^{h}+F_{ah}{}^{c}\ \theta_{b}^{h}+F_{ab0}\ F^{ch}\omega_{h}+F_{ab}{}^{h}\ \theta_{\hat{c}}^{\hat{h}}$$

$$=-F_{ab}{}^{c\,0}\ \omega+(F_{ab0}\ F^{ch}-F_{ab}{}^{ch})\ \omega_{h}$$

$$+((A_{ab}^{fc}+F^{fc}F_{ab})\ F_{fh}-F_{ab}{}^{c}{}_{h}+F_{af}F^{fc}\ F_{bh})\ \omega^{h}$$

$$R_{0ab\hat{c},0}=-F_{ab}{}^{c\,0},$$

$$R_{0ab\hat{c},h}=(A_{ab}^{fc}+F^{fc}F_{ab})\ F_{fh}-F_{ab}{}^{c}{}_{h}+F_{af}F^{fc}\ F_{bh},$$

$$R_{0ab\hat{c},\hat{h}}=F_{ab0}\ F^{ch}-F_{ab}{}^{ch}.$$

4. $(i,j,k,l)=(\hat{a},b,c,\hat{d});\quad t=0,h,\hat{h}$

$$R_{\hat{a}bc\hat{d},0}\ \omega+R_{\hat{a}bc\hat{d},h}\ \omega^{h}+R_{\hat{a}bc\hat{d},\hat{h}}\ \omega^{\hat{h}}=\boldsymbol{d}R_{\hat{a}bc\hat{d}}-R_{0bc\hat{d}}\ \theta_{\hat{a}}^{0}-R_{\hat{h}bc\hat{d}}\ \theta_{\hat{a}}^{\hat{h}}-R_{\hat{a}0c\hat{d}}\ \theta_{b}^{0}-R_{\hat{a}hc\hat{d}}\ \theta_{b}^{h}$$

$$-R_{\hat{a}b0\hat{d}}\ \theta_{c}^{0}-R_{\hat{a}bh\hat{d}}\ \theta_{c}^{h}-R_{\hat{a}bc0}\ \theta_{\hat{d}}^{0}-R_{\hat{a}bc\hat{h}}\ \theta_{\hat{d}}^{\hat{h}}$$

$$=\boldsymbol{d}A_{bc}^{ad}+(\boldsymbol{d}F^{ad})F_{bc}+F^{ad}\ \boldsymbol{d}F_{bc}+F_{bc}{}^{d}\ F^{ah}\omega_{h}$$

$$+(A_{bc}^{hd}+F^{hd}F_{bc})\ \theta_{h}^{a}+F^{ad}{}_{c}\ F_{bh}\omega^{h}+F^{da}{}_{b}\ F_{ch}\omega^{h}$$

$$-(A_{hc}^{ad}+F^{ad}F_{hc})\ \theta_{b}^{h}-(A_{bh}^{ad}+F^{ad}F_{bh})\ \theta_{c}^{h}$$

$$+F_{cb}{}^{a}\ F^{dh}\omega_{h}+(A_{bc}^{ah}+F^{ah}F_{bc})\ \theta_{h}^{d}$$

$$=(A_{bc0}^{ad}+F^{ad0}F_{bc}+F^{ad}\ F_{bc0})\omega+(A_{bch}^{ad}+F^{ad}{}_{h}F_{bc}$$

$$+F^{ad}\ F_{bch}+F^{ad}{}_{c}\ F_{bh}+F^{da}{}_{b}\ F_{ch})\omega^{h}+(A_{bc}^{adh}$$

$$+F^{adh}F_{bc}+F^{ad}\ F_{bc}{}^{h}+F_{bc}{}^{d}\ F^{ah}+F_{cb}{}^{a}\ F^{dh})\omega_{h}$$

$$R_{\hat{a}bc\hat{d},0}=A_{bc0}^{ad}+F^{ad0}F_{bc}+F^{ad}\ F_{bc0},$$

$$R_{\hat{a}bc\hat{d},h}=A_{bch}^{ad}+F^{ad}{}_{h}\ F_{bc}+F^{ad}\ F_{bch}+F^{ad}{}_{c}\ F_{bh}+F^{da}{}_{b}\ F_{ch},$$

$$R_{\hat{a}bc\hat{d},\hat{h}}=A_{bc}^{adh}+F^{adh}F_{bc}+F^{ad}\ F_{bc}{}^{h}+F_{bc}{}^{d}\ F^{ah}+F_{cb}{}^{a}\ F^{dh}.$$

5. $(i,j,k,l)=(a,b,c,d);\quad t=0,h,\hat{h}$

$$R_{abcd,0}\ \omega+R_{abcd,h}\ \omega^{h}+R_{abcd,\hat{h}}\ \omega_{h}=\boldsymbol{d}R_{abcd}-\theta_{a}^{h}\,R_{hbcd}\ -\theta_{b}^{\hat{h}}\ R_{a\hat{h}cd}-\theta_{c}^{h}\ R_{abhd}-\theta_{d}^{h}\ R_{abch}$$

$$=-\boldsymbol{d}(F_{ac}F_{bd}-F_{ad}F_{bc})+(F_{hc}F_{bd}-F_{hd}F_{bc})\ \theta_{a}^{h}$$

$$+(F_{ac}F_{hd}-F_{ad}F_{hc})\ \theta_{b}^{h}+(F_{ah}F_{bd}-F_{ad}F_{bh})\ \theta_{c}^{h}$$

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$$+(F_{ac}F_{bh} - F_{ah}F_{bc})\ \theta_d^h$$

$$R_{abcd,0} = F_{ad0}F_{bc} + F_{ad}F_{bc0} - F_{ac0}F_{bd} - F_{ac}F_{bd0},$$

$$R_{abcd,h} = F_{adh}F_{bc} + F_{ad}F_{bch} - F_{ach}F_{bd} - F_{ac}F_{bdh},$$

$$R_{abcd,\hat{h}} = F_{ad}{}^{h}F_{bc} + F_{ad}F_{bc}{}^{h} - F_{ac}{}^{h}F_{bd} - F_{ac}F_{bd}{}^{h}. \quad \blacksquare$$

4. Curvature Inheritance on C_9 – manifolds:

In this section we study curvature inheritance on C_9 −manifolds.

Theorem 4.1 The C_9 −manifold has a curvature inheritance if and only if there exist an arbitrary scalar function Ψ, such that the following equalities hold:

$$F_{ac0}\, F^{cb} + F_{ac}\, F^{cb0} = \Psi F_{ac}F^{cb}, \tag{15}$$

$$F_{ab00} + 2F^{ch}\, F_{a(h}F_{c)b} = 2\Psi F_{ab0}, \tag{16}$$

$$F_{ab}{}^{c0} + F^{hc}_{\ \ b}\, F_{ha} = 2\Psi F_{ab}{}^{c}, \tag{17}$$

$$A^{ad}_{bc0} + F^{ad0}F_{bc} + F^{ad}F_{bc0} = 2\Psi(A^{ad}_{bc} + F^{ad}F_{bc}), \tag{18}$$

$$F_{ad0}\, F_{bc} + F_{ad}\, F_{bc0} - F_{ac0}\, F_{bd} - F_{ac}\, F_{bd0} = 2\Psi(F_{ad}F_{bc} - F_{ac}F_{bd}). \tag{19}$$

Proof. We can obtain the result from equation (4), Theorems 3.2 and 2.7, Propositions 2.8 and 2.9 and taking:

$$(l,i,j,k,) = (0,0,a,\hat{b}),(0,0,a,b),(\hat{c},0,a,b),(\hat{d},a,b,c),(d,\hat{a},b,c);$$

$$t = 0,h,\hat{h}.$$

1. $(i,j,k,l) = (0,a,\hat{b},0)$

$$R^0_{a\hat{b}0,0} + R^0_{\hat{h}\hat{b}0}\,\xi^{\hat{h}}_{,a} + R^0_{ah0}\,\xi^{h}_{,\hat{b}} = 2\Psi\, R^0_{a\hat{b}0}$$

$$F_{ac0}F^{cb} + F_{ac}F^{cb0} + F_{ah0}F^{hb} + F_{ah}F^{hb0} = 2\Psi F_{ac}F^{cb}$$

$$F_{ac0}F^{cb} + F_{ac}F^{cb0} + F_{ac0}F^{cb} + F_{ac}F^{cb0} = 2\Psi F_{ac}F^{cb}$$

$$2F_{ac0}F^{cb} + 2F_{ac}F^{cb0} = 2\Psi F_{ac}F^{cb}$$

2. $(i,j,k,l) = (0,a,b,0)$

$$R^0_{ab0,0} + R^0_{\hat{h}b0}\,\xi^{\hat{h}}_{,a} + R^0_{a\hat{h}0}\,\xi^{\hat{h}}_{,b} = 2\Psi\, R^0_{ab0}$$

$$F_{ab00} + 2F^{ch}F_{a(h}F_{c)b} = 2\Psi F_{ab0}.$$

3. $(i,j,k,l) = (0,a,b,\hat{c})$

$$R^0_{a\hat{b}0,0} + R^0_{\hat{h}b\hat{c}}\,\xi^{\hat{h}}_{,a} + R^0_{abh}\,\xi^{h}_{,\hat{c}} = 2\Psi\, R^0_{ab\hat{c}}$$

$$F_{ab}{}^{c0} + F^{hc}{}_{b}F_{ha} = 2\Psi F_{ab}{}^{c}.$$

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4. $(i,j,k,l) = (\hat{a}, b, c, \hat{d})$

$$R^{\hat{a}}_{bc\hat{d},0} - R^{\hat{h}}_{bc\hat{d}}\, \xi^{a}_{,\hat{h}} + R^{a}_{\hat{h}c\hat{d}}\, \xi^{\hat{h}}_{,b} + R^{a}_{b\hat{h}\hat{d}}\, \xi^{\hat{h}}_{,c} + R^{a}_{bch}\, \xi^{h}_{,\hat{d}} = 2\Psi\, R^{a}_{bc\hat{d}}$$

$$A^{ad}_{bc0} + F^{ad0}F_{bc} + F^{ad}F_{bc0} = 2\Psi(A^{ad}_{bc} + F^{ad}F_{bc}).$$

5. $(l,i,j,k) = (d, \hat{a}, b, c)$

$$R^{\hat{a}}_{bcd;0} - R^{h}_{bcd}\, \xi^{h}_{,\hat{a}} + R^{\hat{a}}_{\hat{h}cd}\, \xi^{\hat{h}}_{,b} + R^{\hat{a}}_{b\hat{h}d}\, \xi^{\hat{h}}_{,c} + R^{\hat{a}}_{bc\hat{h}}\, \xi^{\hat{h}}_{,c} = 2\Psi\, R^{\hat{a}}_{bcd}$$

$$F_{ad0}F_{bc} + F_{ad}F_{bc0} - F_{ac0}F_{bd} - F_{ac}F_{bd0} = 2\Psi(F_{ad}F_{bc} - F_{ac}F_{bd}). \quad \blacksquare$$

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توارث خاصية الانحناء التناظري في المنطويات من الصنف C_9

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المستخلص:

في هذا البحث ركزنا على تنسر الانحناء الريماني R للمنطويات من الصنف C_9. حيث قمنا بحساب مركبات مشتقة التغاير للتنسر R في الفضاءات ذوات التركيب G. لقد استنتجنا بان هنالك 15 مركبة غير صفرية اساسية من مركبات مشتقة التغاير للتنسر R و بقية المركبات يمكن حسابها باستخدام خاصية التناظر و متطابقات بيانتشي للتنسر R. وفقا لهذه المركبات استطعنا ايجاد الشروط التي تجعل خاصية الانحناء التناظري للتنسر R وراثية في المنطويات من الصنف C_9. تم تلخيص هذه الشروط في 5 معادلات لها دالة عددية اختيارية مشتركة Ψ.