Numerical Solutions of the Second Order Initial Value Problems by Haar Wavelet Method

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Abstract

In this paper, we apply Haar wavelet method for solving seconed order initial value problems of ordinary differential equation. The fundamental idea of Haar wavelet method is to convert the differential equations into a group of algebraic equations, which involves a finite number or variables. We compared this numerical results with the exact solution and other known methods.

Keywords: ordinary differential equation, initial value problems, Haar wavelet method, exact solution**.**

الحل العددي لمسائل قيم ابتدائية من الرتبة الثانية باستعمال طريقة هار الموبجية

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المستخلص

في هذا البحث قمنا بتطبيق طريقة هار الموبجية لحل مسائل قيم ابتائية من الرتبة الثانية للمعادلة التفاضلية. ان الفكرة الاساسية لطريقة هار الموبجية هو تحويل المعادلات التفاضلية الى معادلات جبرية تحتوي على عدد محدود من المتغيرات. ثم قمنا بمقارنة الحل العددي مع الحل الحقيقي وطرائق اخرى معروفة.

1. Introduction

Many field of applications, notably in science and engineering yields initial value problems of second order ordinary differential equations are of the form,

$$y''(x) = \varphi\big(x, y(x), y'(x)\big), \quad y(a) = y_0 , y'(a) = \beta \qquad (1)$$

Many of such problems may not be easily solved analytically , hence numerical schemes are developed to solve (1). These equations are usually reduced to 2 systems of first order ordinary differential equations and numerical methods of first order differential equations are used to solve them. Linear multistep methods are powerful numerical methods for solving differential equations[2] .

2. Derivation of the Haar wavelet method

Form the property of the Haar wavelet Transformation, y'' which is a function of x can be approximated by the Haar wavelet function like wise

$$y''(x) = \varphi(x, y(x), y'(x)) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (2)$$

It is difficult to find the solution $y(x)$ by direct integration the differential equation is nonlinear type or complicated coefficients.
But approximating the $y''(x)$ with Haar wavelet functions, it is quite easier to get $y''(x)$ or $y'(x)$ explicitly in terms of x.
Hence, integrating equation(2), we get ([1],[6]).

$$y'(x) = \int y''(x)dx = \int \left[\sum_{i=1}^{2M} a_i h_i(x)\right] dx = \sum_{i=1}^{2M} \int a_i h_i(x) dx \qquad (3)$$

and

$$y(x) = \int y'(x)dx = \int\int \left[\sum_{i=1}^{2M} a_i h_i(x)\right] dx dx = \sum_{i=1}^{2M} \int\int a_i h_i(x) dx dx \qquad (4)$$

3. Analysis of the Method for Initial Value Problem

Consider the second order Initial Value Problem as

$$y''(x) = \phi(x, y, y') \qquad (5)$$

with initial conditions $y(a) = \alpha, y'(a) = \beta$
If $a = [0,1)$, then

$$y''(x) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (6)$$

$$\int_a^x y''(x')dx' = \int_a^x \sum_{i=1}^{2M} a_i h_i(x')dx' \qquad (7)$$

$$y'(x)-y'(a)=\int_a^x \sum_{i=1}^{2M} a_i h_i(x')dx' \quad (8)$$

$$=\sum_{i=1}^{2M} a_i p_{i,1}(x)-\int_0^a \sum_{i=1}^{2M} a_i h_i(x')dx' \quad (9)$$

$$y(x)-y(a)=\int_a^x \sum_{i=1}^{2M} a_i p_{i,1}(x')dx' \quad (10)$$

$$=(x-a)y'(a)+\sum_{i=1}^{2M} a_i p_{i,2}(x)-\int_0^a \sum_{i=1}^{2M} a_i p_{i,1}(x')dx'-(x-a)A \quad (11)$$

where $$A=\int_0^a \sum_{i=1}^{2M} a_i h_i(x)dx \quad (12)$$

When a = 0, these equations (9) and (11) becomes

$$y'(x)=y'(0)+\sum_{i=1}^{2M} a_i p_{i,1}(x) \quad (13)$$

$$y(x)=y(0)+xy'(0)+\sum_{i=1}^{2M} a_i p_{i,2}(x) \quad (14)$$

Thus to get *y(x)* we have to first find the Haar coefficients unknowns $a_i, i=1,2,.....,2M$ by solving $2M$ system of equations. These $2M$ equations are generated from the relations. i.e. $y''=\phi(x,y,y')$ at $x_j=\frac{j-0.5}{2M}, j=1,2,.....,2M$ which are the collocation points. Now, using equations (9) and (11) in equation (2), we get

$$\sum_{i=1}^{2M} a_i h_i(x_j)=\phi(x_j, y(a)+(x_j-a)(y'(a)+\sum_{i=1}^{2M} a_i p_{i,2}(x_j)-\int_0^a \sum_{i=1}^{2M} a_i p_{i,1}(x_i')\,dx'$$
$$-(x_j-a)A, y'(a)+\sum_{i=1}^{2M} a_i p_{i,1}(x_j)-\int_0^a \sum_{i=1}^{2M} a_i h_i(x_j')\,dx') \quad (15)$$

In this way, we get Haar matrix, as

$$h_i(x) = \begin{cases} 1 & for\ x \in [\alpha, \beta) \\ -1 & for\ x \in [\beta, \gamma) \\ 0 & elsewhere \end{cases} \qquad (16)$$

$H(i,j) = h_i(x_j)$ which has the dimension $N \times N$. For instance, $J=3 \Rightarrow N=16$, then we have

$$H(16,16) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$

We establish an operational matrix for integration via Haar wavelets. The operational matrix of integration is obtained by integrating equation (16) is as,

$$p_{i,1}(x) = \int_0^x h_i(x)\,dx \qquad (17)$$

and

$$p_{i,2}(x) = \int_0^x p_{i,1}(x)\,dx \qquad (18)$$

These integrals can be evaluated by using equation (16) and they are given by

$$p_{i,1}(x) = \begin{cases} x-\alpha & for\ x \in [\alpha, \beta) \\ \gamma - x & for\ x \in [\beta, \gamma) \\ 0 & elsewhere \end{cases} \qquad (19)$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & for\ x \in [\alpha, \beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2 & for\ x \in [\beta, \gamma) \\ \frac{1}{4m^2} & for\ x \in [\gamma, 1) \\ 0 & elsewhere \end{cases} \qquad (20)$$

For instance, $J=3 \Rightarrow N=16$, from equation (19) then we have

$$p_{i,1}(16,16)=\frac{1}{32}\begin{bmatrix}
1&3&5&7&9&11&13&15&17&19&21&23&25&27&29&31\\
1&3&5&7&9&11&13&15&15&13&11&9&7&5&3&1\\
1&3&5&7&7&5&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&5&7&7&5&3&1\\
1&3&3&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&3&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&3&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&3&3&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{bmatrix}$$

and from (20) we get

$$p_{i,2}(16,16)=\frac{1}{2048}\begin{bmatrix}
1&9&25&49&81&121&169&225&289&361&441&529&625&729&841&961\\
1&9&25&49&81&121&169&225&287&343&391&431&463&487&503&511\\
1&9&25&49&79&103&119&127&128&128&128&128&128&128&128&128\\
0&0&0&0&0&0&0&0&1&9&25&49&79&103&119&127\\
1&9&23&31&32&32&32&32&32&32&32&32&32&32&32&32\\
0&0&0&0&1&9&23&31&32&32&32&32&32&32&32&32\\
0&0&0&0&0&0&0&0&1&9&23&31&32&32&32&32\\
0&0&0&0&0&0&0&0&0&0&0&0&1&9&23&31\\
1&7&8&8&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&1&7&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&1&7&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&1&7&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&1&7&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&1&7&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&1&7&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&7
\end{bmatrix}$$

([1],[7]).

4. Convergence Analysis of The Haar Wavelets

In order to check the convergence of the proposed scheme we consider the asymptotic expansion of equation (14) as given below

$$Y(x)=\alpha+\beta x+\sum_{i=0}^{2M-1} a_i p_{i,2}(x) \qquad (21)$$

The error estimation as J^{th} resolution level is

$$\left|Y(x)-y(x)\right|=\left|e_J(x)\right|=\left|\sum_{i=2M}^{\infty} a_i p_{i,2}(x)\right| \tag{22}$$

$$\left\|e_J(x)\right\|^2=\left|\int_{-\infty}^{\infty}\sum_{i=2^{J+1}}^{\infty}\sum_{n=2^{J+1}}^{\infty} a_i a_n \left(p_{i,2}(x)-p_{n,2}(x)\right)dx\right|$$

$$=\left|\sum_{i=2^{J+1}}^{\infty}\sum_{n=2^{J+1}}^{\infty}\int_{-\infty}^{\infty} a_i a_n \left(p_{i,2}(x)-p_{n,2}(x)\right)dx\right|$$

$$\le \sum_{i=2^{J+1}}^{\infty}\sum_{n=2^{J+1}}^{\infty} a_i a_n k_{i,n}$$

where $k_{i,n} \neq (0) \in \Re,\ \forall n = 2^{J+1}, 2^{J+1}+1, \ldots\ldots$ and $k_i = \sup_{n=2^{J+1}}^{\infty} k_{i,n}$.

But $a_n = \int_0^1 2^{\frac{j}{2}} y(x) h\left(2^j x - k\right)dx,\ k = 0,1,2,\ldots\ldots, 2^j - 1$ and $j = 0,1,2,\ldots\ldots, J$

$$h\left(2^j x-k\right)=\begin{cases} 1, & k.2^{-j} \le x < \left(k+\frac{1}{2}\right)2^{-j} \\ -1, & \left(k+\frac{1}{2}\right)2^{-j} \le x < (k+1)2^{-j} \\ 0, & elsewhere \end{cases}$$

Therefore

$$a_n = 2^{\frac{j}{2}}\left(\int_{k.2^{-j}}^{\left(k+\frac{1}{2}\right)2^{-j}} y(x)dx - \int_{\left(k+\frac{1}{2}\right)2^{-j}}^{(k+1)2^{-j}} y(x)dx\right)$$

$$=2^{\frac{j}{2}}\left(\left(\left(k+\frac{1}{2}\right)2^{-j}-k.2^{-j}\right)y(x_1)-\left((k+1)2^{-j}-\left(k+\frac{1}{2}\right)2^{-j}\right)y(x_2)\right)$$

where

$$x_1 \in \left[k.2^{-j}, \left(k+\frac{1}{2} \right) 2^{-j} \right)$$

$$x_2 \in \left[\left(k+\frac{1}{2} \right) 2^{-j}, (k+1) 2^{-j} \right)$$

consequently we have

$$a_n = 2^{\frac{j}{2}-1} \left(y(x_1) - y(x_2) \right)$$

Applying mean value theorem

$$a_n = 2^{-\frac{j}{2}-1} (x_1 - x_2) y'(x)$$

where $x \in \left[k.2^{-j}, (k+1) 2^{-j} \right)$

$$a_n \le 2^{-\frac{j}{2}-1} 2^{-j} D = 2^{\frac{-3j-2}{2}} D \ , \qquad \text{since} \quad y'(x) \le D$$

$$\left\| e_J(x) \right\|^2 \le \sum_{i=2^J+1}^{\infty} a_i k_i \sum_{n=2^J+1}^{\infty} a_n \qquad (23)$$

$$\le \sum_{i=2^J+1}^{\infty} a_i k_i \sum_{n=2^J+1}^{\infty} 2^{\frac{-3j-2}{2}} D$$

$$\le \sum_{i=2^J+1}^{\infty} D a_i k_i \sum_{n=2(J+1)}^{\infty} \sum_{n=2^{\frac{j}{2}}}^{2^{\frac{j}{2}+1}-1} 2^{\frac{-3j-2}{2}}$$

$$\le \sum_{i=2^J+1}^{\infty} D a_i k_i \sum_{n=2(J+1)}^{\infty} 2^{\frac{-3j-2}{2}} 2^{\frac{j}{2}}$$

$$\le \sum_{i=2^J+1}^{\infty} D a_i k_i \sum_{n=2(J+1)}^{\infty} 2^{-j-1}$$

$$\le \sum_{i=2^J+1}^{\infty} D a_i k_i \frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \qquad (24)$$

Similarly $a_i \le D\, 2^{\frac{-3j}{2}-1}$ and $k = \sup_{i=2^{j+1}} (k_i)$.

Therefore

$$\left\|e_J(x)\right\|^2 \le D^2 k \sum_{i=2(J+1)}^{\infty} 2^{\frac{-3j}{2}-1} 2^{\frac{j}{2}} \left(\frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \right)$$

$$\le D^2 k \sum_{i=2(J+1)}^{\infty} 2^{-j-1} \left(\frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \right)$$

$$\le D^2 k \left(\frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \right) \left(\frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \right) \quad (25)$$

$$\left\|e_J(x)\right\|^2 \le D^2 k \left(\frac{2^{-2(J+1)-1}}{1-\frac{1}{4}} \right)^2$$

$$\left\|e_J(x)\right\|^2 \le 2D\sqrt{k} \left(\frac{2^{-2(J+1)}}{3} \right)^2 \quad (26)$$

From equation (26) we can clearly conclude that $\left\|e_J(x)\right\| \to 0$ when $J \to 0$ i.e. error is inversely proportional to the resolution of the Haar wavelet *J*. Thus the proposed composite scheme is convergent [5].

5. Numerical Examples

We have implement our method (15) on are examples of two point initial value problems (1) and (2) , these examples have been solution for different value of the numerical solution, are computed and compared with the exact solution and existing methods. All calculations are implemented by Maple 13.

Example 1 : Consider the initial value problem of the form : ([2],[8])

$$y'' - y' = 0$$

$$y(0) = 0, y'(0) = -1$$

with exact solution $y(x)=1-\exp(x)$.

We first describe the method for second-order linear example with the form:

$$y''(x)-y'(x)=0 \qquad (27)$$

We assume that

$$y''(x)=\sum_{i=1}^{2M} a_i h_i(x) \qquad (28)$$

Integrating equation (28) from 0 to *x*, the derivative $y'(x)$ can be expressed as

$$y'(x)=y'(0)+\sum_{i=1}^{2M} a_i p_{i,1}(x) \qquad (29)$$

from the initial conditions $y'(0)=-1$, substituting this value of $y'(0)$ in equation (29), we obtain

$$y'(x)=\sum_{i=1}^{2M} a_i p_{i,1}(x)-1 \qquad (30)$$

Finally, integrating equation (30) from 0 to *x*, we can express the approximate solution as

$$y(x)=y(0)+\sum_{i=1}^{2M} a_i p_{i,2}(x)-x \qquad (31)$$

from the initial conditions $y(0)=0$, substituting this value of $y(0)$ in equation (31), we get

$$y(x)=\sum_{i=1}^{2M} a_i p_{i,2}(x)-x \qquad (32)$$

we consider only sixteen collocation points. Therefore , $J=3$ and $M=8$. Hence , we need to define $h_i(x), p_{i,1}(x), p_{i,2}(x)$ for $i=1,2,......,16$.

Now, $i=m+k+1$
when $i=1$ then, we get $m=0, k=0$

$$h_1(x)=\begin{cases}1 & for\ x\in[0,1) \\ 0 & elsewher\end{cases}$$

$$p_{1,1}(x)=\begin{cases}x & for\ x\in[0,1) \\ 0 & elsewhere\end{cases}$$

$$p_{1,2}(x)=\begin{cases}\dfrac{x^2}{2} & for\ x\in[0,1)\\ 0 & elsewhere\end{cases}$$

when $i=2$ then, we get $m=1,k=0\Rightarrow\alpha=0,\beta=\frac{1}{2},\gamma=1$

$$h_2(x)=\begin{cases}1 & for\ x\in\left[0,\frac{1}{2}\right)\\ -1 & for\ x\in\left[\frac{1}{2},1\right)\\ 0 & elsewhere\end{cases}$$

$$p_{2,1}(x)=\begin{cases}x-0 & for\ x\in\left[0,\frac{1}{2}\right)\\ 1-x & for\ x\in\left[\frac{1}{2},1\right)\\ 0 & elsewhere\end{cases}$$

$$p_{2,2}(x)=\begin{cases}\frac{1}{2}(x-0)^2 & for\ x\in\left[0,\frac{1}{2}\right)\\ \frac{1}{4.1^2}-\frac{1}{2}(1-x)^2 & for\ x\in\left[\frac{1}{2},1\right)\\ 0 & elsewhere\end{cases}$$

the same way we extract $h_i(x),p_{i,1}(x),p_{i,2}(x)$ for $i=3,4,.....,16$ or matrices $H(16,16),p_{i,1}(16,16),p_{i,2}(16,16)$.

Hence, the given equation becomes

$$y''(x)=\phi(x,y,y')=y'(x) \qquad (33)$$

from equation (15)

i	a_i

$$\sum_{i=1}^{2M} a_i h_i(x_j) = \phi(x_j, y(a) + (x_j - a)(y'(a) + \sum_{i=1}^{2M} a_i p_{i,2}(x_j) - \int_0^a \sum_{i=1}^{2M} a_i p_{i,1}(x_i') \, dx'$$

$$-(x_j - a)A, y'(a) + \sum_{i=1}^{2M} a_i p_{i,1}(x_j) - \int_0^a \sum_{i=1}^{2M} a_i h_i(x_j') \, dx')$$

from equation (33) we get

$$\phi(x,y,y') = \left[y'(0) + \sum_{i=1}^{16} a_i p_{i,1}(x_j) - \int_0^a \sum_{i=1}^{16} a_i h_i(x')dx' \right] \quad (34)$$

as $a = 0$ (given), thus

$$\phi(x,y,y') = \left[y'(0) + \sum_{i=1}^{16} a_i p_{i,1}(x_j) \right] \quad (35)$$

from the initial conditions $y'(0) = -1$, substituting this value of $y'(0)$ inequation (35), we obtain

$$\sum_{i=1}^{16} a_i h_i(x_j) = \sum_{i=1}^{16} a_i p_{i,1}(x_j) - 1$$

$$\sum_{i=1}^{16} a_i h_i(x_j) - \sum_{i=1}^{16} a_i p_{i,1}(x_j) + 1 = 0$$

$$\sum_{i=1}^{16} a_i \left(h_i(x_j) - p_{i,1}(x_j) \right) + 1 = 0$$

$$a_1\left(h_1(x_j) - p_{1,1}(x_j)\right) + a_2\left(h_2(x_j) - p_{2,1}(x_j)\right) + ... + a_{16}\left(h_{16}(x_j) - p_{16,1}(x_j)\right) + 1 = 0$$
(36)

we put the values of $h_i, p_{i,1}$ in equation (36) where $i = 1,2,.......,16$. When $x_j = \dfrac{j-0.5}{16}, j = 1,2,........,16$, we get a system of equations and solving this system , we get given in the following table Haar coefficients

1	-1.719167349
2	0.4211877559
3	0.1614596871
4	0.2662453765
5	0.07096319412
6	0.09112596480
7	0.1170175830
8	0.1502657861
9	0.03329864724
10	0.03773384688
11	0.04275979110
12	0.04845516390
13	0.05490912954
14	0.06222272848
15	0.07051045922
16	0.07990207086

We using these values of a_i and $p_{i,2}(x)$ in the relation

$$y(x) = \sum_{i=1}^{16} a_i p_{i,2}(x) - x$$

solving the above equation we get the approximate solution .We compare the Haar Wavelet solution with the exact solution at different grid points. Table (1) represents the comparison when M=8 and table (2) represents the comparison when M= 16.
Table(3) shows compared of our method with other methods([2],[8]). Figure (1) shows the exact and numerical solution for $M = 16$, the same way we solve the following example.

Example 2 : Consider the initial value problem of the form :[3]

$$y'' + 8y' + 16y = 0$$

$$y(0) = 1, y'(0) = -12$$

with exact solution $y(x) = (1-8x)\exp(-4x)$.
The numerical solution of the example (2) are presented in Tables (4)and (5) that contains results for our method for different value $M = 8$ and 16 respectively , Table(6) shows compared of our method with other methods([3],[4]). Figure (2) shows the exact and numerical solution for $M = 16$.

Table 1:The numerical solution and exact solution of example (1) at $M = 8$.

x	Exact Solution	*Haar solution*	*Error*
0.1	-.105170918	-.1052081165	3.71985E-05
0.2	-.221402758	-.2214799988	7.72408E-05
0.3	-.349858808	-.3499934172	1.346092E-04
0.4	-.491824698	-.4920177788	1.930808E-04
0.5	-.648721271	-.6489897966	2.685256E-04
0.6	-.822118800	-.8224769073	3.581073E-04
0.7	-1.013752707	-1.014208055	4.55348E-04
0.8	-1.225540928	-1.226125370	5.84442E-04
0.9	-1.459603111	-1.460322094	7.18983E-04

Table 2:The numerical solution and exact solution of example (1) at $M = 16$.

x	Exact Solution	*Haar solution*	*Error*
0.1	-.105170918	-.1051796469	8.7289E-06
0.2	-.221402758	-.2214293383	2.65803E-05
0.3	-.349858808	-.3498919424	3.3134E-05
0.4	-.491824698	-.4918736363	4.89383E-05
0.5	-0.64872127	-.6487883685	6.70975E-05
0.6	-.822118800	-.8222073468	8.85468E-05
0.7	-1.013752707	-1.013867211	1.14504E-04
0.8	-1.225540928	-1.225686134	1.45206E-04

0.9	-1.459603111	-1.459783899	1.80788E-04

Table3: Comparison of our method with other methods for example (1). when $M = 8$.

x	Error in Haar W. Method	*Error of* [Ehigie 2013]	*Error of* [*Yahaya* 2009]
0.1	3.71E-05	3.38E-04	8.79E-05
0.2	7.72E-05	7.95E-04	3.27E-04
0.3	1.34E-04	1.40E-03	2.22E-03
0.4	1.93E-04	2.16E-03	4.86E-03
0.5	2.68E-04	3.13E-03	9.10E-03
0.6	3.58E-04	4.33E-03	1.44E-02
0.7	4.55E-04	5.79E-03	2.15E-02
0.8	5.84E-04	7.56E-03	2.99E-02
0.9	7.18E-04	9.70E-03	4.03E-02

Table4:The numerical solution and exact solution of example (2) at $M = 8$.

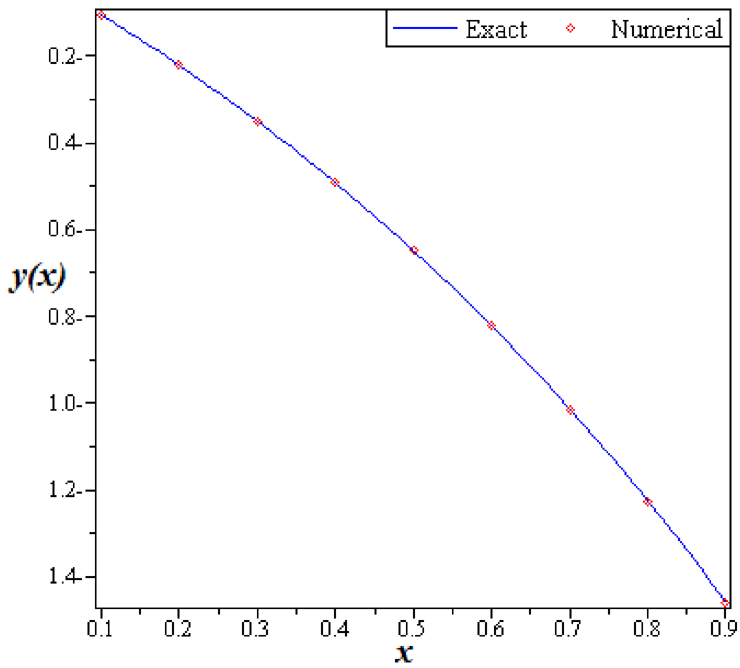
x	Exact Solution	*Haar solution*	*Error*
0.1	0.1340640092	0.1270722581	6.9917511E-03
0.2	-.2695973785	-.274498311	4.9009325E-03
0.3	-.4216718967	-.423424617	1.7527203E-03
0.4	-.4441723396	-.442541152	1.6311876E-03
0.5	-.4060058496	-.402449148	3.5567016E-03
0.6	-.3447282225	-.340247056	4.4811665E-03
0.7	-.2797262881	-.275090712	4.6355761E-03
0.8	-.2201159015	-.215832353	4.2835485E-03
0.9	-.1694070792	-.165699637	3.7074422E-03

Table 5:The numerical solution and exact solution of example (2) at $M = 16$.

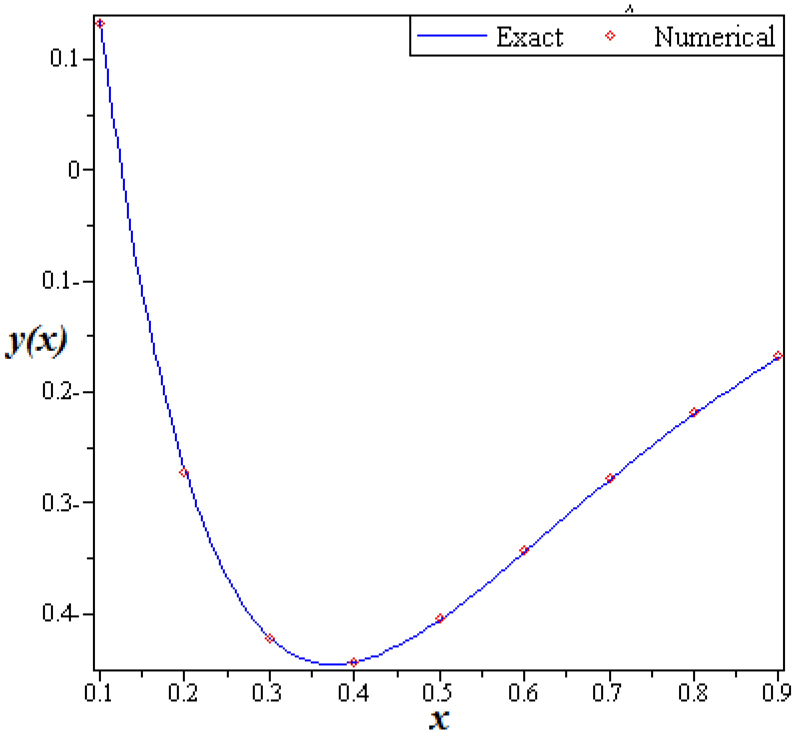
x	Exact Solution	*Haar solution*	*Error*
0.1	0.1340640092	0.1324516985	1.6123107E-03
0.2	-.2695973785	-.271628809	2.0314305E-03
0.3	-.4216718967	-.422068906	3.970093E-04
0.4	-.4441723396	-.443796215	3.761246E-04
0.5	-.4060058496	-.405122777	8.830726E-04
0.6	-.3447282225	-.343611265	1.1169575E-03
0.7	-.2797262881	-.2785750224	1.1512657E-03
0.8	-.2201159015	-.2190470257	1.0688758E-03
0.9	-.1694070792	-.1684663368	9.407424E-04

Table 6: Comparison of our method with other methods for example (2). when $M = 8$

x	Error in Haar W.Method	*Error of* BBDF2[3]	*Error of* *BDFVS*[4]
10^{-2}	5.0765E-04	1.9115E-03	1.1749E-00
10^{-4}	5.8E-08	1.1411E-04	1.2581E-02
10^{-6}	0.0E-00	4.6212E-06	1.2791E-04



Figure(1):Comparison of exact and numerical solutions of example (1) for $M = 16$.



Figure(2):Comparison of exact and numerical solutions of example (2)for $M = 16$.

6. Conclusion

In this paper, Haar wavelet method are used to develop a class of numerical methods for solving initial value problems. The numerical results obtained by using the method described in this study give acceptable results. The comparison with analytical solution shows that Haar wavelet gives better results with less computational cost: it is due to the sparsity of the transform matrix and small number of wavelet coefficients. Wavelet provides excellent results for small and large values of m .

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